Chapter 2

Optimal Control Theory

2.1 Optimal Controller

* Optimal control technique considers not only performance spec. and stability, but also control energy requirment.
* In the contrast with the classical control techniques, the optimal control techniques meet the performance specifications by using the performance criterion function.

The most typical performance criterion for the SISO-LTI System is

$$J_0 = \frac{1}{2}\int_0^\infty \{x^T Q x + u^T R u\} dt \qquad \rightarrow \text{ Quadratic Criterion}$$

subject to

$$\dot{x} = Ax + bu, \; x(0) = x_0 \qquad \rightarrow \text{ Linear System}$$

Our objective here is to find the control intput $u(t), \; 0 \leq t \leq T$ that minimizes J_0 and this controller is called the Linear-Quadratic Regulator (LQR). For the above performance criterion, we normally choose $Q \geq 0$, $R > 0$.

* For Set-point control rather than regulation, we use

$$J_1 = \frac{1}{2}\int_0^\infty \{y(t) - r(t))^T Q(y(t) - r(t)\} + u^T(t)Ru(t)\}dt$$

* Q and R dictate the relative penalty on the regulation capability and energy cost. If Q is relatively larger than R, then we put more weight on regulation and if R is larger than Q, we respond more sensitively to the energy cost.

(Ex) Practical Set-point Control

In the case of SISO system, $J = \int_0^\infty \{(y-r)^2 + \rho u^2\}dt$ (Q and R is const, $\therefore \rho = \frac{R}{Q}$)

Solving the above control problem with J_0 (proof given later), we have the optimal control value: $u^*(t) = -k^T \cdot x^*(t)$, which is in the feedback form of the optimal state $x^*(t)$. This is very important.

The k value here is represented as $k^T = R^{-1}b^T s_\infty$, where S_∞(symmetric) satisfies:

$$A^T S_\infty + S_\infty A + Q - S_\infty b R^{-1} b^T S_\infty = 0 \text{ and } S_\infty > 0 \qquad \cdots \quad (1)$$

Note here that S_∞ exists when the given LTI System is C.C & C.O. The performance criterion $J_0 = \int_0^\infty (x^T qx + u^T Ru)dt$ is particularly called the infinite horizon criterion. If performance criterion is given by $J_T = \frac{1}{2}x^T(T)\bar{S}x(T) + \frac{1}{2}\int_0^T (x^T qx + u^T Ru)dt$, it is called the finite horizon criterion.

Solving the finite horizon problem, we have $u^*(t) = -l^T(t)x(t)$ and $k^T(t) = R^{-1}b^T S(t)$, where $S(t)$ satisfies

$$-\dot{S}(t) = A^T S(t) + S(t)A + Q - S(t)bR^{-1}b^T S(t) \qquad \text{with } S(T) = \bar{S} \quad \cdots \quad (2)$$

In the above, equation (1) is called the Algebraic Riccati Equation(ARE) and equation (2) is called the Matrix Riccati Equation(MRE).

(Ex) Given $G(s) = \frac{1}{s^2}$ with state eqn. $\begin{cases} \dot{x} = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{b} u \\ y = \underbrace{(1\ 0)}_{c^T} x \end{cases}$

Here, find the optimal controller that minimizes $J_0 = \int_0^\infty \{y^2 + \rho u^2\}dt$.

(Solution): First, it is evident that

$$J_0 = \int_0^\infty \{\underbrace{x^T \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{y^T} \underbrace{(1\ 0)x}_{y} + \rho u^2\}dt$$

$$= \int_0^\infty \{x^T \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{Q \geqq 0} x + \underbrace{\rho u^2}_{R>0}\}dt$$

Note here that $Q \geqq 0$ iff $\forall x \neq 0 \ \exists \ x^T Q x \geqq 0$

and $Q > 0$ iff $\forall x \neq 0 \ \exists \ x^T Q x > 0$

So, if $Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $x = \begin{pmatrix} p \\ q \end{pmatrix}$, $x^T Q x = p^2 \geq 0$

$\therefore \ Q \geqq 0$

Since Q and R are given, we solve the ARE (letting $S_\infty = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}$), which becomes

$$A^T S_\infty + S_\infty A + Q - S_\infty b R^{-1} b^T S_\infty = 0$$

$$\rightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix} + \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$- \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rho^{-1}(0\ 1) \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 0 & 0 \\ s_1 & s_2 \end{pmatrix} + \begin{pmatrix} 0 & s_1 \\ 0 & s_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \rho^{-1} \begin{pmatrix} s_2^2 & s_2 s_3 \\ s_2 s_3 & s_3^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} -\rho^{-1}s_2^2 + 1 & -\rho^{-1}s_2 s_3 + s_1 \\ -\rho^{-1}s_2 s_3 + s_1 & -\rho^{-1}s_3^2 + 2s_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\therefore s_2^2 = \rho \ \rightarrow \ s_2 = \sqrt{\rho}, \ -\sqrt{\rho}$

And from $2s_2 - \rho^{-1}s_3^2 = 0$, we have $s_3^2 = 2\rho s_2$. But notice that if $s_2 = -\sqrt{\rho}$, s_3 has no solution. Therefore, s_2 must be $\sqrt{\rho}$ and in this case, we have $s_3 = \pm\sqrt{2} \cdot \rho^{\frac{3}{4}}$.

But since $s_\infty > 0$, we have $s_3 = \sqrt{2} \cdot \rho^{\frac{3}{4}}$. Finally, $s_1 = \rho^{-1}s_2 s_3 = \frac{1}{\rho}\sqrt{\rho} \cdot \sqrt{2} \cdot \rho^{\frac{3}{4}} = \sqrt{2}\rho^{\frac{1}{4}}$

$$\therefore\ s_\infty = \begin{pmatrix} \sqrt{2}\rho^{\frac{1}{4}} & \sqrt{\rho} \\ \sqrt{\rho} & \sqrt{2}\rho^{\frac{3}{4}} \end{pmatrix}$$

Next, $k^T = R^{-1}b^T s_\infty = \frac{1}{\rho}(0\ 1)s_\infty = (\frac{1}{\sqrt{\rho}},\ \sqrt{2} \cdot \rho^{-\frac{1}{4}})$ and

$u(t) = -k^T x(t) = -\frac{1}{\sqrt{\rho}}x_1(t) - \sqrt{2}\rho^{-\frac{1}{4}}x_2(t)(Ans.)$

Using the above state feedback controller, we have

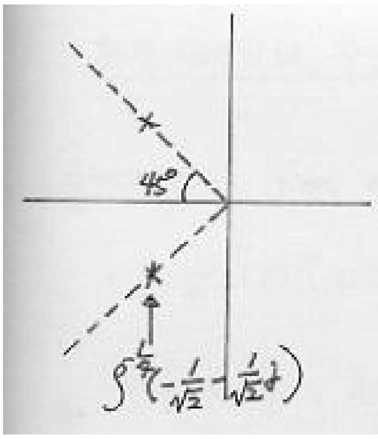
$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u = \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} (\frac{1}{\sqrt{\rho}},\ \sqrt{2} \cdot \rho^{-\frac{1}{4}}) \right] \cdot x$$

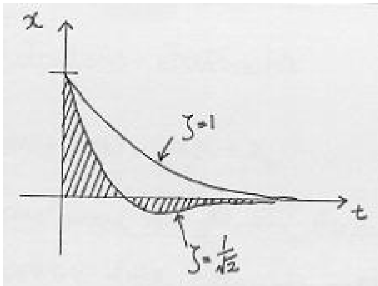
$$= \underbrace{\begin{pmatrix} 0 & 1 \\ -\frac{1}{\sqrt{\rho}} & -\sqrt{2}\rho^{-\frac{1}{4}} \end{pmatrix}}_{A_c} x$$

The eigenvalues of this closed-loop system becomes:

$$\det(sI - A_c) = s^2 + \sqrt{2}\rho^{-\frac{1}{4}}s + \frac{1}{\sqrt{\rho}} = 0$$

$$\therefore \; \underline{s = \rho^{-\frac{1}{4}}(-\frac{1}{\sqrt{2}} \pm j\frac{1}{\sqrt{2}})}$$





<u>This optimal controller has the damping ratio (ζ) $\frac{1}{\sqrt{2}}$ independent of ρ values.</u>

As in the above example, when we find the controller that minimizes $J_0 = \int_0^\infty (y^2 + \rho u^2)dt$, the resulting closed loop eigenvalues turn out be the stable roots of $1 + \rho^{-1}G(s)G(-s) = 0$ <u>(Symmetric Root Locus(SRL))</u>.

In the case of above example, $G(s) = \frac{1}{s^2}$ and $G(-s) = \frac{1}{s^2}$
$\therefore 1 + \rho^{-1} \cdot \frac{1}{s^4} = 0 \;\rightarrow\; s^4 = -\frac{1}{\rho} \quad \therefore \; s = \rho^{-\frac{1}{4}}(\pm\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}j)$
The stable roots are of course $s = \rho^{-\frac{1}{4}}(-\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}j)$. Clearly, these roots are the roots of $(s + \rho^{-\frac{1}{4}} \cdot \frac{1}{\sqrt{2}})^2 = -\frac{1}{\sqrt{\rho}} \cdot \frac{1}{2}$ and of $s^2 + \sqrt{2} \cdot \rho^{-\frac{1}{4}}s + \rho^{-\frac{1}{2}} = 0$. These

roots are also the eigenvalues of the matrix:

$$\begin{pmatrix} 0 & 1 \\ -\frac{1}{\sqrt{\rho}} & -\sqrt{2}\rho^{-\frac{1}{4}} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} (k_1 \;\; k_2) \text{ and } \begin{cases} k_1 = \frac{1}{\sqrt{\rho}} \\ k_2 = \sqrt{2}\rho^{-\frac{1}{4}} \end{cases}$$

Derivation of optimal controller

Objective: find an optimal controller $u^*(t)$ that minimizes

$$J_T = \frac{1}{2}x^T(T)\bar{S}x(T) + \frac{1}{2}\int_0^T \{x^T(\tau)Qx(\tau) + u^T(\tau)Ru(\tau)\}d\tau$$

subject to $\dot{x}(t) = Ax(t) + bu(t), \quad x(0) = x_0$

Derivation: This is the constrained minimization. The most typical method of solving this problem is to use the Lagrange multiplier technique and transform the original problem to the problem as follows:

$$J_T(u^*, x^*) = J'_T(u^*, x^*, \lambda^*) = J_T(u^*, x^*) + \int_0^T \lambda^{T*}(\tau)[-\dot{x}^*(\tau) + Ax^*(\tau) + bu^*(\tau)]d\tau$$

Note here that J'_T does not assume any constraints and is smooth with respect to x, u, λ (continuously differential). On the other hand, J'_T satisfies:

$$J'_T(u^*, x^*, \lambda) \;\leq\; J'_T(u^*, x^*, \lambda^*) \;\leq\; J'_T(u, x, \lambda^*)$$

Clearly, we have $\frac{\partial J'_T}{\partial x} = 0$, $\frac{\partial J'_T}{\partial u} = 0$, $\frac{\partial J'_T}{\partial \lambda} = 0$ at (u^*, x^*, λ^*).

$$\therefore \;\; \frac{\partial J'_T}{\partial u}|_{(x^*,u^*,\lambda^*)} = U^*T \cdot R + \lambda^{*T} \cdot b = 0 \;\rightarrow\; \underline{u^* = -R^{-1} \cdot B^T \cdot \lambda^*} \;\; (\textit{control eqn.})$$

$$\frac{\partial J'_T}{\partial x}|_{(x^*,u^*,\lambda^*)} = x^{*T} \cdot Q + \lambda^{*T} \cdot A + \dot{\lambda}^*T = 0$$

$$\rightarrow\; \underline{\dot{\lambda}^* = -A^T \cdot \lambda^* - Q \cdot x^*} \;\; (\textit{costate eqn.}) \;\forall\; 0 < t < T$$

(Here we used the fact that

$$\int_0^T \lambda^T \cdot \dot{x} dt = \lambda^T \cdot x|_0^T - \int_0^T \dot{\lambda}^T \cdot x dt = \lambda^T(T)x(T) - \lambda^T(0)x(0) - \int_0^T \dot{\lambda}^T(\tau)x(\tau)d\tau)$$

$$\frac{\partial J'_T}{\partial \lambda}|_{(x^*,u^*,\lambda^*)} = \underline{-\dot{x}^* + Ax^* + bu^* = 0} \ \ (state\ eqn.)\ \forall\ 0 < t \leq T$$

Finally

$$\frac{\partial J'_T}{\partial x}|_{(x^*(T),u^*(T),\lambda^*(T))} = \underline{x^T(T) \cdot \bar{S} - \lambda^{*T}(T) = 0} \ \ (final\ condition\ for\ \lambda(t))$$

Therefore, (x^*, λ^*) are computed from:

$$\begin{cases} \dot{x}^* = Ax^* + bu^* = Ax^* - bR^{-1}b^T\lambda^* & with\ x*(0)\ =\ x_0 \\ \dot{\lambda}^* = -A^T \cdot \lambda^* - Qx^* & and\ \lambda^*(T)\ =\ \bar{S}x^*(T) \end{cases} \quad \cdots\cdots\ (1)$$

This is called the <u>two-point boundary value problem.</u>

At this point, by using the so-called <u>sweep method</u>, we know that $\lambda^*(t) = S(t)x^*(t)$.

In fact, $X(t)$, $\bigwedge(t)$ be the solutions of (1) subject to $X(T)\ =\ I$ & $\lambda(T)\ =\ \bar{S}$. Then by linear superposition,

we have $x^*(t) = X(t)x^*(T)$ & $\lambda^*(t) = \Lambda(t)x^*(T)$, and hence

$$\underline{\lambda^*(t) = \Lambda(t)X^{-1}(t)x^*(t) = S(t)x^*(t)} \qquad \cdots\cdots\ (2)$$

Differentiating (2), we have $\dot{\lambda}^*(t) = \dot{S}(t)x^*(t) + S(t)\dot{x}^*(t)$

But from (1), we have $-A^T\lambda^*(t) - Qx^*(t) = \dot{S}(t)x^*(t) + S(t)(Ax^*(t) + bu^*(t))$

$$\therefore [\dot{S}(t) + S(t)A + A^TS(t) + Q - S(t)bR^{-1}b^TS(t)] \cdot x^*(t) = 0$$

This equation holds irrespective of $x^*(t)$. So it follows that

$$-\dot{S}(t) = S(t)A + A^TS(t) + Q - S(t)bR^{-1}b^TS(t)\ \&\ S(T) = \bar{S} \quad (MRE)$$

and $u^*(t) = -\underbrace{R^{-1}b^TS(t)}_{k^T(t)}x^*(t) \qquad \cdots\cdots\ (3)$

Nest, if (A, b) is C.C and (A, Q) is C.O, $J_T \to J_0$ as $T \to \infty$ and $\dot{S}(t) \to$

0 & $S_\infty > 0$ as $t \to \infty$.

Then we have

$$S_\infty A + A^T S_\infty + Q - S_\infty b R^{-1} b^T S_\infty = 0 \quad (ARE)$$

and S_∞ is unique.

Also, $u^*(t) = -\underbrace{R^{-1} b S_\infty}_{k^T} \cdot x^*(t) \qquad \cdots\cdots \; (4)$

Therefore, with Q & R given, we can determine k^T.

For (3), $J_T^{min} = J_T(u^*, x^*) = \underline{\frac{1}{2} x_0^T S(0) x_0}$

For (4), $J_0^{min} = J_0(u^*, x^*) = \underline{\frac{1}{2} x_0^T S_\infty x_0}$

Prove these two statements.(H.W. #2.1)

2.2 Kalman Filter

$$\text{plant: } \begin{cases} \dot{x}(t) = Ax(t) + bu(t) + \underbrace{w(t)}_{system\ noise} \\ y(t) = c^T x(t) + \underbrace{v(t)}_{sensornoise} \end{cases}$$

Here $E(w(t)) = E(v(t)) = 0,\ E(x(0)) = x_0$

$E(w^T(t) \cdot w(\tau)) = Q_0 \delta(t - \tau),\ E(v^T(t) \cdot v(\tau)) = R_0 \delta(t - \tau)$

where δ isa delta function. Also,

$$E(w^T(t) v(\tau)) = 0,\ E(x(0) v^T(t)) = E(x(0) w^T(t)) = 0$$

(note that Q_0 & R_0 are symmetric)

For this plant, the observer generating $\hat{x}(t)$, the estimate of $x(t)$, from the output $y(t)$ is given by

$$\dot{\hat{x}}(t) = A\hat{x}(t) + bu(t) + \underbrace{l(t)}_{time-varying}(y(t) - \hat{y}(t))$$

where $l(t) = P(t)cR_0^{-1}$, $P(t)$ is symmetric and

$$-\dot{P}(t) = AP(t) + P(t)A^T + Q_0 - P(t)cR_0^{-1}P(t) \quad with\ P(0) = P_0\ >\ 0$$

This is finite time Kalman Filter.
If (A, c) is C.O and A, Q_0 is C.C, $\dot{P}(t) \to 0$ and $P(t) \to P_\infty\ >\ 0$ as $t \to \infty$.
Then we have

$$AP_\infty + P_\infty A^T + Q_0 - P_\infty cR_0^{-1}c^T P_\infty = 0$$

and $l(t) = l_\infty \to P_\infty cR_0^{-1}$. The observer in this case becomes:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + bu(t) + l_\infty(y(t) - \hat{y}(t))$$

This is the infinite-time or steady state Kalman Filter.

2.3 Linear Quadratic Gaussian(LQG) Controller:

$$\text{plant:} \begin{cases} \dot{x}(t) = Ax(t) + bu(t) + \underbrace{w(t)}_{system\ noise} \\ y(t) = c^T x(t) + \underbrace{v(t)}_{sensornoise} \end{cases}$$

$$with\ x(0)\ =\ x_0,\ cov(x(0)) = P_0,\ E(w(t)) = E(v(t)) = 0$$
$$E(w(t)w^T(\tau)) = Q_0\delta(t-\tau),\ E(v(t)v^T(\tau)) = R_0\delta(t-\tau)$$
$$E(w(t)v^T(\tau)) = 0\ (Q_0,\ R_0\ symmetric,\ Q_0 \geq 0,\ R_0 \geq 0)$$
$$E(x(0)v^T(t)) = E(x(0)w^T(t)) = 0$$

Criterion: $J = E\{\frac{1}{2}x^T(T)\bar{S}x(T) + \frac{1}{2}\int_0^T\{x^T(\tau)Qx(\tau) + u^T(\tau)Ru(\tau)\}d\tau\}$

Objective: find an optimal controller $u^*(t)$ that minimizes the above criterion

Solution: The optimal controller is given as $u^*(t) = -k^T(t)\hat{x}(t)$, where

(i) $k^T(t) = R^{-1}b^TS(t)$ and

$-\dot{S}(t) = A^TS(t) + S(t)A + Q - S(t)bR^{-1}b^TS(t)$ with $S(T) = \bar{S}$

(ii) $\hat{x}(t)$ is determined from $\dot{\hat{x}}(t) = A\hat{x} + bu(t) + l(t)(y(t) - \hat{y}(t))$ with $l(t) = P(t)cR_0^{-1}$

and $-\dot{P}(t) = AP(t) + P(t)A^T + Q_0 - P(t)cR_0^{-1}P(t)$ with $P(0) = P_0 > 0$

Now, when the steady-state Kalman Filter is used to minimize

Criterion:

$$J_\infty = \lim_{T\to\infty}\frac{1}{2T}E\int_0^T\{x^T(t)Qx(t) + u^T(t)Ru(t)\}dt,$$

then the optimal controller is: $u^*(t) = -k^T(t)\hat{x}(t)$, where $K^T = R^{-1}b^TS_\infty$

and $A^TS_\infty + S_\infty A + Q - S_\infty bR^{-1}b^TS_\infty = 0$

and the optimal observer is $\dot{\hat{x}}(t) = A\hat{x}(t) + bu(t) + l_\infty(y(t) - \hat{y}(t))$,

where $l_\infty = p_\infty cR_0^{-1}$ and $AP_\infty + P_\infty A^T + Q_0 - P_\infty cR_0^{-1}c^TP_\infty = 0$.

(Ex) 1st order system: $\begin{cases} \dot{x} = \frac{1}{2}x + u + w & E[w(t)w^T(t)] = q\delta(t-\tau) \\ y = x + v & E[v(t)v^T(t)] = r\delta(t-\tau) \end{cases}$

$$J_\infty = \lim_{T\to\infty}\frac{1}{2T}E\int_0^T(ax^2 + bu^2)dt$$

$$0 = s_\infty - a + \frac{1}{b}S_\infty^2 \Rightarrow S_\infty = -\frac{b}{2} + \sqrt{\frac{b^2}{4} + ab} \Rightarrow k_\infty^T = \sqrt{\frac{1}{4} + \frac{a}{b}} - \frac{1}{2}$$

$$0 = -P_\infty + q - \frac{1}{r}P_\infty^2 \Rightarrow P_\infty = -\frac{r}{2} + \sqrt{\frac{r^2}{4} + qr} \Rightarrow l_\infty = \sqrt{\frac{1}{4} + \frac{q}{r}} - \frac{1}{2}$$

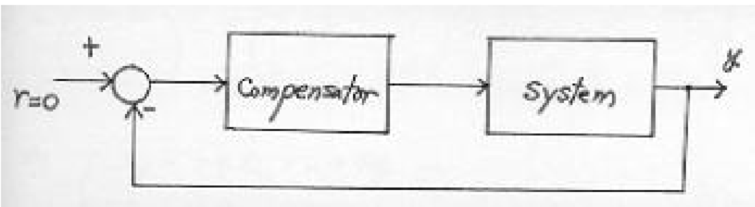
Hence, $\mathrm{u} = -[\sqrt{\frac{1}{4} + \frac{\mathrm{a}}{\mathrm{b}}} - \frac{1}{2}]\hat{\mathrm{x}}$ with $\dot{\hat{\mathrm{x}}} = -\sqrt{\frac{1}{4} + \frac{\mathrm{a}}{\mathrm{b}}}\hat{\mathrm{x}} + (\sqrt{\frac{1}{4} + \frac{\mathrm{q}}{\mathrm{r}}} - \frac{1}{2})(\mathrm{y} - \hat{\mathrm{x}})$

(Ex) <u>2nd order system:</u>
$$\begin{cases} \dot{x} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u + w \\ y = (0 \;\; 1)x + v, \quad where\ for\ all\ t, \tau \geq 0, \end{cases}$$

$$E[w(t)w^T(t)] = \begin{pmatrix} 16 & 0 \\ 0 & 0 \end{pmatrix} \delta(t-\tau),\ E[w(t)] = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$E[v(t)v^T(t)] = \delta(t-\tau),\ E[v(t)] = 0,\ and\ E[w(t)v^T(\tau)] = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Now let's determine the compensator transfer function that minimizes $J_\infty = \lim_{T\to\infty} \frac{1}{2T} E \int_0^T (ax^2 + bu^2)dt\ k_\infty$. The transfer function can be computed from the optimal control gain k_∞ and the Kalman filter.



Here,

$$A = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad c = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad Q_0 = \begin{pmatrix} 16 & 0 \\ 0 & 0 \end{pmatrix} \quad R_0 = 1$$
$$Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = cc^T \quad R = 1$$

Since $u = -k_\infty^T \cdot x = -R^{-1} b^T S_\infty x$, let's define S_∞ as $\begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}$.

Then, form $A^T S_\infty + S_\infty A + Q - S_\infty b R^{-1} b^T S_\infty = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, we have

$$\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix} + \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$- \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot 1 \cdot (1 \ \ 0) \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}$$
$$= \begin{pmatrix} 2s_2 - s_1^2 & s_3 - s_2 - s_1 s_2 \\ s_3 - s_2 - s_1 s_2 & -2s_3 + 1 - s_2^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \ \Rightarrow \ S_\infty \begin{pmatrix} \sqrt{3} - 1 & 2 - \sqrt{3} \\ 2 - \sqrt{3} & -3 + 2\sqrt{3} \end{pmatrix}$$
$$\therefore \ k^T = R^{-1} b^T S_\infty = 1 \cdot (1 \ \ 0) S_\infty = \underline{(\sqrt{3} - 1, \ \ 2 - \sqrt{3})}$$

Also from $\dot{\hat{x}} = A\hat{x} + bu + l_\infty (y - \hat{y}), \ l_\infty = P_\infty c R_0^{-1}$ and

$$A P_\infty + P_\infty A^T + Q_0 - P_\infty c R_0^{-1} c^T P_\infty = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix} + \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 16 & 0 \\ 0 & 0 \end{pmatrix}$$
$$- \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot 1 \cdot (0 \ \ 1) \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}$$
$$= \begin{pmatrix} 16 - p_2^2 & p_1 - p_2 - p_2 p_3 \\ p_1 - p_2 - p_2 p_3 & 2p_2 - 2p_3 - p_3^2 \end{pmatrix} \ \Rightarrow \ \therefore \ P_\infty = \begin{pmatrix} 12 & 4 \\ 4 & 2 \end{pmatrix} \text{ and } \mathrm{l}_\infty = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

Therefore, optimal observer(s.s. Kalman filter) is

$$\dot{\hat{x}} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \hat{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u + \begin{pmatrix} 4 \\ 2 \end{pmatrix} (y - \hat{y}) = \begin{pmatrix} 0 & -4 \\ 1 & -3 \end{pmatrix} \hat{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u + \begin{pmatrix} 4 \\ 2 \end{pmatrix} y$$

$$\Rightarrow \begin{cases} \dot{\hat{x}}_1 = -4\hat{x}_2 + u + 4y \\ \dot{\hat{x}}_2 = \hat{x}_1 - 3\hat{x}_2 + 2y \end{cases}$$

Next, compensator is $C(s) = k_\infty^T (sI - A + bk_\infty^T + l_\infty c^T)^{-1} l_\infty$.

First,

$$sI - A + bk_\infty^T + l_\infty c^T = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} \sqrt{3}-1 & 2-\sqrt{3} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 4 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} s+\sqrt{3}-1 & 6-\sqrt{3} \\ -1 & s+3 \end{pmatrix}$$

$$(sI - A + bk_\infty^T + l_\infty c^T)^{-1} = \frac{1}{(s+3)(s+\sqrt{3}-1)+6-\sqrt{3}} \begin{pmatrix} s+\sqrt{3}-1 & 6-\sqrt{3} \\ -1 & s+3 \end{pmatrix}$$

$$= \frac{1}{s^2+(2+\sqrt{3})s+3+2\sqrt{3}} \begin{pmatrix} s+3 & \sqrt{3}-6 \\ 1 & s+\sqrt{3}-1 \end{pmatrix}$$

$$\therefore\ C(s) = \frac{(\sqrt{3}-1,\ \ 2-\sqrt{3})}{s^2+(2+\sqrt{3})s+3+2\sqrt{3}} \begin{pmatrix} s+3 & \sqrt{3}-6 \\ 1 & s+\sqrt{3}-1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$= \underline{\frac{2\sqrt{3}s+4}{s^2+(2+\sqrt{3})s+3+2\sqrt{3}}} \qquad (Ans)$$

2.4 Optimal control for discrete time systems

* Find a set of control inputs $\{u(k),\ k=0,\ 1,\ 2,\ \cdots,\ N-1\}$ that minimizes

$$J_N = \frac{1}{2}x^T(N)\bar{S}x(N)+\frac{1}{2}\sum_{k=0}^{N-1}\{x^T(k)Qx(k)+u^T(k)Ru(k)\} \quad Q=Q^T\geq 0,\ R=R^T\geq 0$$

subject to $x(k+1)=Ax(k)+bu(k)$ with $x(0)=x_0$

(Solution) When we use the Lagrange multiplier techniques as in the continuous time system, ti follows that

$$J_N(u^*,\ x^*)=J'_N(u^*,\ x^*,\ \lambda^*)=J_N(u^*,\ x^*)+\sum_{k=0}^{N-1}[\lambda^{*T}(k+1)(-x^*(k+1)+Ax^*(k)+bu^*(k))]$$

Now, we must find $(u^*,\ x^*,\ \lambda^*)$ that makes

$$J'_N(u^*,\ x^*,\ \lambda)\leq J'_N(u^*,\ x^*,\ \lambda^*)\leq J'_N(u,\ x,\ \lambda^*)$$

Since this is the unconstrained minimization problem with quadratic performance criterion, $\frac{\partial J'_N}{\partial x}=0,\ \frac{\partial J'_N}{\partial u}=0$ and $\frac{\partial J'_N}{\partial \lambda}=0$ at $(u^*,\ x^*,\ \lambda^*)$.

$$\therefore\ \frac{\partial J'_N}{\partial u(k)}|_{(u^*,\ x^*,\ \lambda^*)}=u^*(k)\cdot R+\lambda^{*T}(k+1)\cdot b=0 \qquad \text{for } 0\leq \text{k} \leq \text{N}-1$$

$$\rightarrow\ \underline{u^*(k)=-R^{-1}b^T\lambda^*(k+1)}\ \ (control\ eqn.)$$

$$\frac{\partial J'_N}{\partial x(k)}|=x^{*T}(k)Q+\lambda^{*T}(k+1)\cdot A-\lambda^{*T}=0 \qquad \text{for } 0\leq \text{k} \leq \text{N}-1$$

$$\rightarrow\ \underline{\lambda^*(k)=A^T\lambda^*(k+1)+Qx^*(k)}\ \ (costate\ eqn.)$$

$$\frac{\partial J'_N}{\partial \lambda(k+1)}|_{(u^*,\ x^*,\ \lambda^*)}=-x^*(k+1)+Ax^*(k)+bu^*(k)=0 \qquad \text{for } 0\leq \text{k} \leq \text{N}-1$$

$$\rightarrow\ \underline{x^*(k+1)=Ax^*(k)+bu^*(k)}\ \ (state\ eqn.)$$

$$\frac{\partial J'_N}{\partial x(N)}|_{(u^*,\ x^*,\ \lambda^*)}=x^{*T}(N)\bar{S}-\lambda^{*T}(N)=0$$

$$\rightarrow\ \underline{\lambda^*(N)=\bar{S}\cdot x^*(N)}\ \ (final\ condition)$$

Therefore, $(x^*,\ \lambda^*)$ can be determined from

$$\begin{cases} x^*(k+1) = Ax^*(k) + bu^*(k) = Ax^*(k) - bR^{-1}b^T\lambda^*(k+1) & x^*(0) = x_0 \\ \lambda^*(k) = A^T\lambda^*(k+1) + Qx^*(k) & \lambda^*(N) = \bar{S}\cdot x^*(N) \end{cases} \quad \cdots (1)$$

Now, here $X(k),\ \Lambda(k)$ be the solution fo (1) with boundary conditions $x(N) = I$ & $\lambda(N) = \bar{S}$. In this case, $x^*(k) = X(k)x^*(N)$ & $\lambda^*(k) = \Lambda(k)x^*(N)$, and hence

$$\underline{\lambda^*(k) = \Lambda(k)X^{-1}(k)x^*(k) = S(k)x^*(k)} \quad \cdots\cdots (2)$$

$$\begin{aligned} \therefore\ u^*(k) &= -R^{-1}b^T\lambda^*(k+1) = -R^{-1}b^TS(k+1)x^*(k+1) \\ &= -R^{-1}b^TS(k+1)(Ax^*(k) + bu^*(k)) \\ \rightarrow\ u^*(k) &= -\underbrace{(R + b^TS(k+_1)b)^{-1}\cdot b^TS(k+1)A}_{k^T(k)\ (Kalman\ Gain)}x^*(k) \quad \cdots\cdots (3) \end{aligned}$$

On the other hand, from $\lambda^*(k) = A^T\lambda^*(k+1) + Qx^*(k)$, we have

$$S(k)x^*(k) = A^TS(k+1)x^*(k+1) + Qx^*(k) = A^TS(k+1)[Ax^*(k) + bu^*(k)] + Qx^*(k)$$

$$\therefore\ [S(k) - A^TS(k+1)A + A^TS(k+1)b(R + b^TS(k+1)b)^{-1}b^TS(k+1)A - Q]x^*(k) = 0$$

Since this equation always holds for any values of $x^*(k)$, we obtain

$$S(k) = A^T[s(k+1) - S(k+1)b(R+b^TS(k+1)b)^{-1}b^TS(k+1)]A + Q \quad (MRE)$$

with $S(N) = \bar{S}$ When $(A,\ b)$ is C.C and $(A,\ Q)$ is C,O,

$$J_N \rightarrow J_\infty = \frac{1}{2}\sum_{k=1}^{\infty}\{x^T(k)Qx(k) + u^TRu(k)\}$$

and $S(k) \rightarrow S_\infty > 0$ as $N \rightarrow \infty$. In this case, the MRE turns out to be

$$S_\infty = A(S_\infty - S_\infty b(R + b^TS_\infty b)^{-1}b^TS_\infty)A + Q \quad (ARE)$$

and the optimal controller becomes

$$u^*(k) = -\underbrace{(R + b^T S_\infty b)^{-1} \cdot b^T S_\infty A}_{k_\infty^T} \cdot x^*(k) \cdots\cdots (4)$$

In the case of (3), we have $J_N^{min} = J_N(u^*, x^*) = \frac{1}{2} x_0^T S(0) x_0$, and
in the case of (4), we have $J_0^{min} = J_0(u^*, x^*) = \frac{1}{2} x_0^T S_\infty x_0$
Prove these two cases (H.W. #2.2)
(Ex) When sampling the system $G(s) = \frac{1}{s^2}$ with sample time $T = 0.1$ (sec),

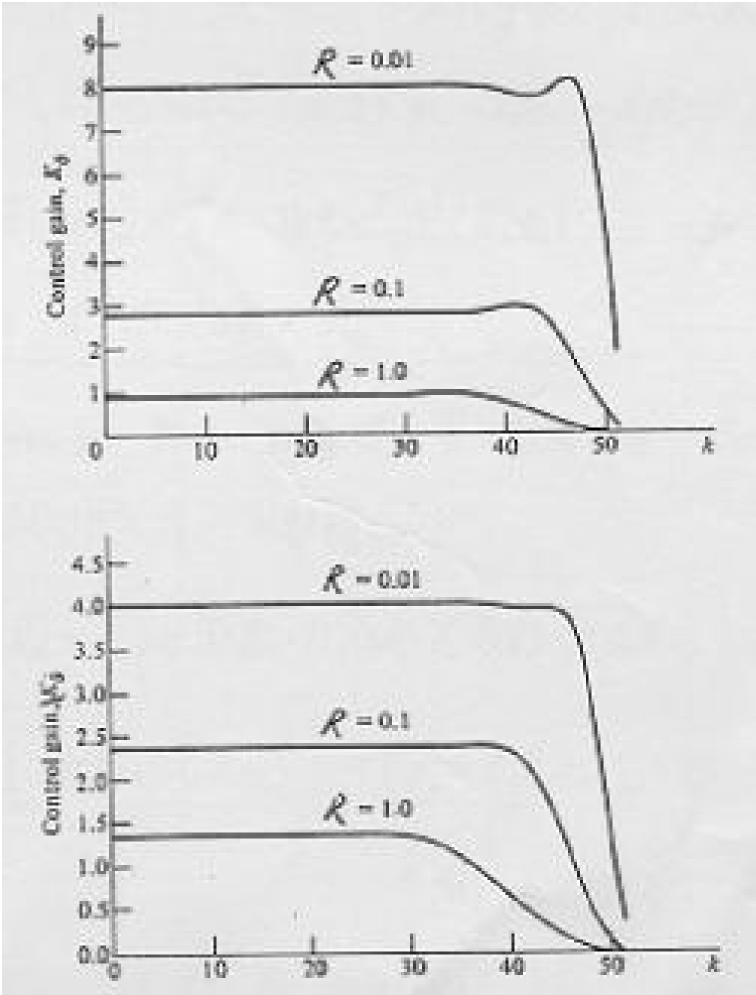
$$G(z) = \frac{T^2}{2} \cdot \frac{z+1}{(Z-1)^2} = \frac{0.01}{2} \cdot \frac{z+1}{(z-1)^{@}}$$

The controllable canonical form realization corresponding to the above transfer function is $\begin{cases} \dot{x} = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} \frac{T^2}{2} \\ T \end{pmatrix} u \\ y = (1 \;\; 0) x \end{cases}$

Let $Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ (or $Q = CC^T$) and change $R = 0.01,\ 0.1\ \&\ 1$. Now, the optimal gain $K = \begin{pmatrix} k_\theta \\ k_{\dot{\theta}} \end{pmatrix}$ that minimizes the performance criterion

$$J = \frac{1}{2} x^T(51) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x(51) + \sum_{k=0}^{50} \{ \frac{1}{2} x^T(k) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x(k) + \frac{1}{2} u^T(k) R u(k) \}$$

with up to $N = 51$ (steps) is given as:



2.5 Kalman Filter for discrete time system

$$\text{plant: } \begin{cases} x(k+1) = Ax(k) + bu(k) + \underbrace{w(k)}_{system\ noise} \\ y(k) = c^T x(k) + \underbrace{v(k)}_{sensornoise} \end{cases}$$

Here,

$$E(x(0)) \; = \; x_0, \; cov(x(0)) = P_0, \; E(w(k)) = E(v(k)) = 0$$
$$E(w(k)w^T(l)) = Q_0\delta(k-l), \; E(v(k)v^T(l)) = R_0\delta(k-l)$$

$$E(w(k)v^T(l)) = 0 \quad Q_0,\ R_0\ symmetric,\ \delta(k-l) = \begin{cases} 1\ when\ k = l \\ 0\ when\ k \neq l \end{cases}$$

$$E(x(0)v^T(k)) = E(x(0)w^T(k)) = 0$$

In this case, the observer estimating $x(k)$ from $y(k)$ is

$$\underbrace{\hat{x}(k+1) = A\hat{x}(k) + bu(k) + l(k)(y(k) - c^T\hat{x}(k))}_{linear\ observer} \quad \cdots\cdots\ (2)$$

and the estimation error $\tilde{x}(k) = x(k) - \hat{x}(k)$ satisfies the equation:

$$\tilde{x}(k+1) = A\tilde{x}(k) + w(k) - l(k)(y(k) - c^T\hat{x}(k)) = (A - l(k)c^T)\tilde{x}(k) + w(k) - l(k)v(k)$$

with $E(\tilde{x}(k+1)) = (a - l(k)c^T)E(\tilde{x}(k))$. Note here that for all k with

$$\underbrace{E(\tilde{x}(k)) = 0\ if\ E(\hat{x}(0)) = x_0}_{unbiased} \quad \cdots\cdots\ (3)$$

Kalman Filter is the Best Linear Unbiased Estimate(BLUE) satisfying (2) and (3). Also, the error variance $P(k) = E[(\tilde{x}(k) - E\tilde{x}(k))(\tilde{x}(k) - E\tilde{x}(k))^T]$ satisfies

$$P(k+1) = E[\tilde{x}(k+1)\tilde{x}(k+1)^T] = (A - l(k)c^T)P(k)(A - l(k)C^T)^T + Q_0 + l(k)R_0l^T(k)$$

Now, the objective is to minimize the variance $P(k+1)$.

First, note here that $P(k+1) \geq 0$ whenever $P(k) \geq 0$.
So, the criterion is to minimize $\alpha^T P(k+1)\alpha$ for arbitrary $\alpha \neq 0$.

$$\begin{aligned} \alpha^T P(k+1)\alpha \ &= \ \alpha^T[AP(k)A^T + Q_0 - l(k)c^TP(k)A^T \\ &\quad - AP(k)cl^T(k) + l(k)(c^TP(k)c + R_0)l^T(k)]\alpha \\ &= \ \alpha^T[AP(k)A^T + Q_0 - AP(k)c(R_0 + c^TP(k)c)^{-1}c^TP(k)A^T]\alpha \\ &\quad + \alpha^T[\{l(k) - AP(k)c(R_0 + c^TP(k)c)^{-1}\}(R_0 + c^TP(k)c)\{l(k) \\ &\quad - AP(k)C(R_0 + c^TP(k)C)^{-1}\}^T]\alpha \end{aligned}$$

Here, the first term is independent of $l(k)$ and the second term is nonnegative. So, if we take $l(k)$ so as to make the second term to zero, then this $l(k)$ minimizes $\alpha^T P(k+1)\alpha$.

$$\therefore\ \underline{l(k) = AP(k)c(R_0 + c^T P(k)c)^{-1}} \quad \text{and}$$
$$\underline{P(k+1) = AP(k)A^T + Q_0 - AP(k)c(R_0 + c^T P(k)c)^{-1}c^T P(k)A^T}$$

with $P(0) = P_0 > 0$. If $(A,\ c)$ is C.O and (A, Q_0) is C.C, $P(k) \to P_\infty > 0$ as $k \to \infty$. In this case, it follows that

$$P_\infty = AP\infty A^T + Q_0 - AP_\infty c(R_0 + c^T P_\infty c)^{-1} c^T P_\infty A^T$$

and $l_\infty = AP_\infty c(R_0 + c^T P_\infty c)^{-1}$. And the optimal observer becomes

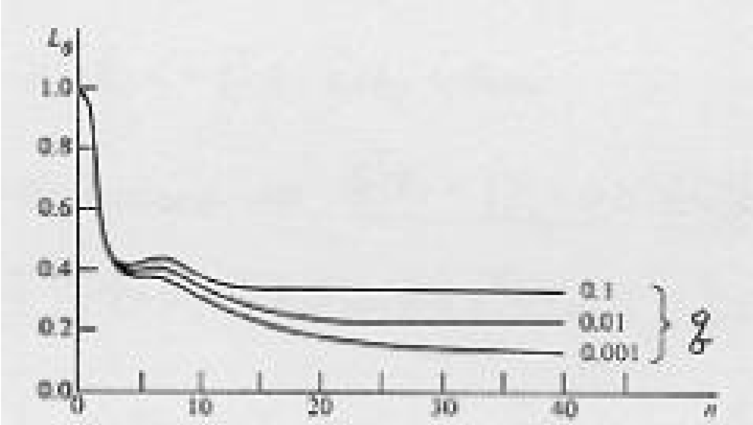
$$\underline{\hat{x}(K+1) = A\hat{x}(k) + bu(t) + l_\infty(y(k) - \hat{y}(k)}$$

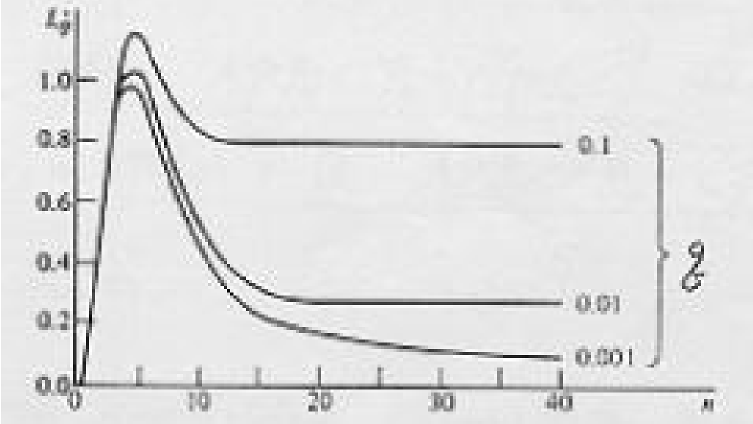
(Ex) Let's use the system as used to determine the optimal controller:

$$\begin{cases} \dot{x} = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} \frac{T^2}{2} \\ T \end{pmatrix} u \\ y = (1 \ \ 0)x \end{cases}$$

Let $R_0 = 0.1$ (deg^2) and $Q_0 = \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}$ with $q = 0.001,\ 0.01,\ 0.1\ (deg^2/sec^4)$.

Then the optimal time varying observer(Kalman Filter) gain $= \begin{pmatrix} L_\theta \\ L_{\dot{\theta}} \end{pmatrix}$ can be depicted as:





2.6 LQG(Linear Quadratic Gaussian) Controller for discrete time systems

plant: $\begin{cases} x(k+1) = Ax(k) + bu(k) + w(k) \\ y(k) = c^T x(k) + v(k) \end{cases}$

with $\mathrm{E(x(0)) = x_0,\ cov(x(0)) = P_0,\ E(w(k)) = E(v(k)) = 0}$

$E(w(k)w^T(l)) = Q_0\delta(k-l),\ E(v(k)v^T(l)) = R_0\delta(k-l)$

$E(w(k)v^T(l)) = 0 \quad (Q_0,\ R_0\ symmetric,\ Q_0 \geq 0,\ R_0 \geq 0)$

$E(x(0)v^T(k)) = E(x(0)w^T(k)) = 0$

Criterion: $J_N = E\{\frac{1}{2}x^T(N)\bar{S}x(N) + \frac{1}{2}\sum_{k=0}^{N-1}\{x^T(k)Qx(k) + u^T(k)Ru(k)\}\}$
$(Q = Q^T \geq 0,\ R = R^T \geq 0)$

Objective: find an optimal controller $u^*(k)$ that minimizes the above criterion

Solution: $u^*(k) = -k^T(k) \cdot \hat{x}(k)$ where

(i) $k(k)$ determined as $k^T(k) = (R + b^T S(k+1)b)^{-1} b^T S(k+1)A$ and

$S(k) = A^T[S(k+1) - S(k+1)b(R + b^T S(k+1)b)^{-1} b^T S(k+1)]A + Q$ with $S(N) = \bar{S}$.

(ii) $\hat{x}(k)$ is determined from $\hat{x}(k+1) = A\hat{x}(k) + bu(k) + l(k)(y(k) - \hat{y}(k)$.
where $l(k) = AP(k)c(R_0 + c^T P(k)c)^{-1}$ and
$P(k+1) = AP(k)A^T + Q_0 - AP(k)c(R_0 + c^T P(k)c)^{-1} c^T P(k)A^T$
with $P(0) = P_0 > 0$

Next, when the steady-state Kalman Filter is used to minimize

$$J_\infty = \lim_{N\to\infty} \frac{1}{2N} E\{\sum_{k=0}^{N} x^T(k)Qx(k) + u^T(k)Ru(k)\}$$

, then the optimal controller is:
$u^*(k) = -k^T\hat{x}(k)$ where $k^T = (R + b^T S_\infty b)^{-1} b^T S_\infty A$ with

$$S_\infty = A^T(S_\infty - S_\infty b(R + b^T S_\infty b)^{-1} b^T S_\infty)A + Q$$

and $\hat{x}(k+1) = A\hat{x}(k) + bu(k) + l_\infty y(k) - \hat{y}(k)$, where
$l_\infty = AP_\infty c(R_0 + c^T P_\infty c)^{-1}$ and

$$P_\infty = AP_\infty A^T + Q_0 - AP_\infty c(R_0 + c^T P_\infty c)^{-1} c^T P_\infty A^T$$