2012 American Control Conference
Fairmont Queen Elizabeth, Montréal, Canada
June 27-June 29, 2012

Controller Design for a Class of Nonlinear Systems with State and Input Bounds

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***Abstract*—In this paper, we propose a bounded controller for nonlinear systems in strict feedback form that are constrained with state and input bounds. We apply the backstepping technique with barrier Lyapunov function to develop an appropriate bounded controller. Then we propose the sufficient conditions for the above nonlinear systems to satisfy the state and input bound constraints and to make the control parameters region larger. The computer simulation establishs the validity of the proposed method.**

I. INTRODUCTION

Many real systems have constraints because of physical stoppages, saturation limits, safety specifications, etc. If we simply apply the conventional controller to this system without taking these constraints into account, the system performance may be significantly degraded or the system may even go unstable. So, during the past few decades, a number of research results have been reported that deal with these issues.

Various control strategies have been proposed to tackle state and input constraints in linear systems. Key approaches include the set invarience method [1], the admissible set control method [2], the reference governer method [3], and the linear model predictive control technique [4]. In nonlinear systems, many techniques have also been proposed that include the invariance control method [5], the nonlinear reference governer methods [6], [7], and the nonlinear model predictive control techniques [8], [9].

However, many of these methods required heavy on-line computation effort. To overcome the on-line computational burden, barrier Lyapunov function based backstepping control techniques have recently been suggested. The backstepping controller is a recursive procedure that combines the choice of Lyapunov function with the design of feedback control, most of the involved computations are carried out off-line, and the on-line computational burden is significantly reduced. The value of the barrier Lyapunov function goes to infinity whenever its states approach the boundary, thereby satisfying the state bounds. By designing the controller so that the time derivative of the barrier Lyapunov function continues to be negative, we can keep the barrier Lyapunov function bounded, and the state bounds are satisfied. This controller that satisfies the state bounds have been used for system in Brunosky form [10], strict feedback form [11], [12], and output feedback form [13].

The controller that satisfies the state and input bounds are also suggested in [14]. However, the developed controller does not satisfy the given input bounds, but ensures only input boundedness. In this paper, we propose a bounded controller that satisfies the given input bound requirements. Assuming that the control Lyapunov function (CLF) is known for control-affine nonlinear system, we can stabilize this system with the bounded controller [15] while satisfying input bounds. For control-affine nonlinear system in strict feedback form, the CLF can be obtained by using the backstepping technique with the barrier Lyapunov function. Using this CLF and the developed bounded controller, we stabilize the system, in this paper, subject to state and input bounds.

The paper is organized as follows. In Section II, we introduce the nonlinear system in strict feedback form, barrier Lyapunov function, and the backstepping controller example based on a barrier Lyapunov function. We introduce the bounded controller and propose a bounded controller using the backstepping technique with barrier Lyapunov function, from which bounded controller input is generated and applied as a backstepping controller input in Section III. Simulation results are described in Section IV, and finally, conclusion is drawn in Section V.

II. BACKSTEPPING CONTROLLER BASED ON BARRIER LYAPUNOV FUNCTION

A. Problem Formulation and Preliminary

In this paper, we use the following notations. R^n is n-dimensional Euclidean space, and we denote $\bar{x}_i = [x_1, x_2, ..., x_i]^T$, $\bar{z}_i = [z_1, z_2, ..., z_i]^T$, and $\bar{y}_d^i = [y_d^{(1)}, y_d^{(2)}, ..., y_d^{(i)}]^T$. We omit time variable t from time-dependent variables, because we deal with time-invariant system throughout this paper.

Consider a nonlinear system with full state bound

This research was supported by the MKE (The Ministry of Knowledge Economy), Korea, under the ITRC (Information Technology Research Center) support program supervised by the NIPA (National IT Industry Promotion Agency) (NIPA-2010-(C-1090-1021-0006)) Also, this work was supported by the Brain Korea 21 Project in 2012.

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constraints in strict feedback form:

$$\begin{aligned}\dot{x}_i &= f_i(\bar{x}_i) + g_i(\bar{x}_i)x_{i+1}, \quad i = 1, ..., n-1 \\ \dot{x}_n &= f_n(\bar{x}_n) + g_n(\bar{x}_n)u, \\ y &= x_1, \end{aligned} \tag{1}$$

where $x_i \in R$ is the state which is required to stay in $|x_i| < k_{c_i}$, with k_{c_i} being a positive constant, for all $i = 1, ..., n$, and $u \in R$, $y \in R$ are respectively the control input and output. For the controller design in the sequel, we introduce the following assumptions and definitions.

Assumption 1. The function f_i and g_i are smooth and bounded, that is, infinitely differentiable and $|f_i(\bar{x}_i)| \le \bar{f}_i$ and $|g_i(\bar{x}_i)| \le \bar{g}_i$ for all $i = 1, ..., n$ where $\bar{f}_i$ and $\bar{g}_i$ are positive constants. Moreover, there exists a positive constant g_0 such that $0 < g_0 \le |g_i(\bar{x}_i)|$ for all $i = 1, ..., n$.

Assumption 2. The desired trajectory y_d and its time derivatives $y_d^{(i)} = \frac{d^{(i)}y_d}{dt^{(i)}}$ are bounded, that is, $|y_d| \le Y_0$ and $|y_d^{(i)}| \le Y_i$ for all $i = 1, ..., n$ where $Y_0, ..., Y_n$ are positive constants.

Definition 1. A barrier Lyapunov function is a scalar function of $V(x)$, defined with respect to the system $\dot{x} = f(x, u)$ on an open region D containing the origin. This function is positive definite and continuously differentiable, and $V(x) \to \infty$ as x approaches the boundary of D.

Definition 2. Let $V : R^n \to R$ be a continuously differentiable, proper, and positive definite function. $V(x, u)$ is a control Lyapunov function (CLF) for the system $\dot{x} = f(x, u)$ if, for all $x \ne 0$, there exists a u such that $\dot{V}(x, u) < 0$.

B. Backstepping Controller Based on Barrier Lyapunov Function

To help better understand the backstepping controller based on the barrier Lyapunov function, We start with this controller applied to the second-order nonlinear system with full state bound constraints in strict feedback form as follows:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2, \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)u, \\ y &= x_1, \end{aligned} \tag{2}$$

where f_1, f_2 and g_1, g_2 satisfy Assumption 1, and $x_1, x_2 \in R$ are the states, with x_1, x_2 required to satisfy $|x_1| < k_{c_1}$, $|x_2| < k_{c_2}$, with k_{c_1}, k_{c_2} being positive constants.

Step 1. Denote $z_1 = x_1 - y_d$ and $z_2 = x_2 - \alpha_1$ where α_1 is a stabilizing function to be designed. Choose a barrier Lyapunov function candidate as

$$V_1 = \frac{1}{2}\log\frac{k_{b_1}^2}{k_{b_1}^2 - z_1^2}. \tag{3}$$

and take the time derivative on both sides of equation (3) to have

$$\dot{V}_1 = \frac{z_1\dot{z}_1}{k_{b_1}^2 - z_1^2} = \frac{z_1(f_1 + g_1(z_2 + \alpha_1) - \dot{y}_d)}{k_{b_1}^2 - z_1^2}. \tag{4}$$

Choosing α_1 as

$$\alpha_1 = \frac{1}{g_1}(-f_1 - k_1z_1 + \dot{y}_d), \tag{5}$$

where k_1 is a positive constant and substituting (5) into (4), we have

$$\dot{V}_1 = -k_1z_1^2 + \frac{g_1z_1z_2}{k_{b_1}^2 - z_1^2}. \tag{6}$$

The second term of the right side of (6) is to be considered in the following step.

Step 2. By augmenting V_1, we next choose another barrier Lyapunov function candidate as

$$V_2 = V_1 + \frac{1}{2}\log\frac{k_{b_2}^2}{k_{b_2}^2 - z_2^2}. \tag{7}$$

Then the time derivative of V_2 is given as

$$\dot{V}_2 = -k_1z_1^2 + \frac{g_1z_1z_2}{k_{b_1}^2 - z_1^2} + \frac{z_2(f_2 + g_2u - \dot{\alpha}_1)}{k_{b_2}^2 - z_2^2}. \tag{8}$$

Choosing the control law as

$$u = \frac{1}{g_2}(-f_2 + \dot{\alpha}_1 - (k_{b_2}^2 - z_2^2)k_2z_2 - \frac{k_{b_2}^2 - z_2^2}{k_{b_1}^2 - z_1^2}g_1z_1), \tag{9}$$

where k_2 is a positive constant, and substituting (9) into (8), we have $\dot{V}_2 = -k_1z_1^2 - k_2z_2^2$.

Extending the result to the nth order system, we apply the backstepping controller using barrier Lyapunov function to nth order system (1).

Denoting the error variables $z_1 = x_1 - y_d$ and $z_i = x_i - \alpha_{i-1}$, $i = 2, ..., n$, we can construct the barrier Lyapunov function as

$$V = \sum_{k=1}^{n} V_i, \quad V_i = \frac{1}{2}\log\frac{k_{b_i}^2}{k_{b_i}^2 - z_i^2}, \quad i = 1, ..., n \tag{10}$$

where $k_{b_1} = k_{c_1} - Y_1$ and k_{b_i}, $i = 2, ..., n$, are the positive parameters to be chosen in Theorem 1. We design α_i as follows:

$$\alpha_1 = \frac{1}{g_1}(-f_1 - (k_{b_1}^2 - z_1^2)k_1z_1 + \dot{y}_d), \tag{11}$$

$$\begin{aligned}\alpha_i =& \frac{1}{g_i}(-f_i + \dot{\alpha}_{i-1} - (k_{b_i}^2 - z_i^2)k_iz_i \\ &- \frac{k_{b_i}^2 - z_i^2}{k_{b_{i-1}}^2 - z_{i-1}^2}g_{i-1}z_{i-1}), \quad i = 2, ..., n \end{aligned} \tag{12}$$

$$u = \alpha_n, \tag{13}$$

where k_i, $i = 1, ..., n-1$, are positive constants to be determined in Theorem 1, k_n is any positive constant, and $\dot{\alpha}_i$ is represented as

$$\dot{\alpha}_i = \sum_{j=1}^{i}\frac{\partial \alpha_i}{\partial x_j}(f_j + g_jx_{j+1}) + \sum_{j=1}^{i}\frac{\partial \alpha_i}{\partial y_d^j}y_d^{j+1}, \quad i = 1, ..., n-1 \tag{14}$$

Then we have

$$\dot{z}_1 = -(k_{b_1}^2 - z_1^2)k_1 z_1 + g_1 z_2,$$
$$\dot{z}_i = -(k_{b_i}^2 - z_i^2)k_i z_i - \frac{k_{b_i}^2 - z_i^2}{k_{b_{i-1}}^2 - z_{i-1}^2} g_{i-1} z_{i-1} + g_i z_{i+1}, \quad i = 2, ..., n-1$$
$$\dot{z}_n = -(k_{b_n}^2 - z_n^2)k_n z_n - \frac{k_{b_n}^2 - z_n^2}{k_{b_{n-1}}^2 - z_{n-1}^2} g_{n-1} z_{n-1} \quad (15)$$

and the time derivative of V yields

$$\dot{V} = -\sum_{j=1}^{n} k_j z_j^2 < 0. \quad (16)$$

As a result, the closed loop system becomes asymptotically stable.

Theorem 1. Consider the above closed loop system (15) and let $\xi = [k_1, ..., k_{n-1}, k_{b_2}, ..., k_{b_n}]^T$ be the control parameters. Given $k_{c_i} > 0$, $i = 2, ..., n$, if we design the control parameters ξ such that the following conditions are satisfied,

$$C1) \quad X_i(\xi) < k_{c_i}, \quad i = 2, ..., n \quad (17)$$
$$C2) \quad |z_i(0)| < k_{b_i}, \quad i = 1, ..., n \quad (18)$$

where

$$X_i(\xi) = \max |x_i| = \max_{\bar{z}_i \in \Omega_{\bar{z}_i}, \bar{y}_d^{i-1} \in \Omega_{\bar{y}_d^{i-1}}} |z_i + \alpha_{i-1}| \quad (19)$$

, $i = 2, ..., n$, and

$$\Omega_{\bar{z}_i} = \{\bar{z}_i \in R^i | \Pi_{k=1}^n (k_{b_k}^2 - z_k^2(0)) \leq \frac{\Pi_{k=1}^n k_{b_k}^2}{\Pi_{k=1}^i k_{b_k}^2} \cdot \Pi_{k=1}^i (k_{b_k}^2 - z_k^2)\} \quad (20)$$
$$\Omega_{\bar{y}_d^i} = \{\bar{y}_d^i \in R^i | y_d^{(j)} \leq Y_j, j = 1, ..., i\} \quad (21)$$

then the controller designed with the above design parameters ξ makes the closed loop system that satisfies:
(i) every state $x_i(t)$ remains in the compact set $x_i \leq X_i < k_{c_i}$, $i = 1, ..., n$.
(ii) every error state $z_i(t)$, $i = 1, ..., n$, converges to the origin within the compact set $\Omega_{\bar{z}_n}$.

Proof: (i) Since $\dot{V} < 0$, we have

$$\frac{1}{2}\log\left(\Pi_{k=1}^n \frac{k_{b_k}^2}{k_{b_k}^2 - z_k^2}\right) \leq \frac{1}{2}\log\left(\Pi_{k=1}^n \frac{k_{b_k}^2}{k_{b_k}^2 - z_k^2(0)}\right), \quad (22)$$

and since $0 \leq \frac{k_{b_j}^2}{k_{b_j}^2 - z_j^2} \leq 1$ for all $i \leq j \leq n$, it follows from (22) that

$$\frac{1}{2}\log\left(\Pi_{k=1}^i \frac{k_{b_k}^2}{k_{b_k}^2 - z_k^2}\right) \leq \frac{1}{2}\log\left(\Pi_{k=1}^n \frac{k_{b_k}^2}{k_{b_k}^2 - z_k^2(0)}\right) \quad (23)$$

for all $i \leq n$. Multiplying 2 and taking exponential on both sides of the equation, and rearranging the equation, we have

$$\Pi_{k=1}^n (k_{b_k}^2 - z_k^2(0)) \leq \frac{\Pi_{k=1}^n k_{b_k}^2}{\Pi_{k=1}^i k_{b_k}^2} \cdot \Pi_{k=1}^i (k_{b_k}^2 - z_k^2). \quad (24)$$

Then from (20), the domain of $X_i(\xi)$ for $\bar{z}_i$ becomes

$$\Omega_{\bar{z}_i} = \{\bar{z}_i \in R^i | \Pi_{k=1}^n (k_{b_k}^2 - z_k^2(0)) \leq \frac{\Pi_{k=1}^n k_{b_k}^2}{\Pi_{k=1}^i k_{b_k}^2} \cdot \Pi_{k=1}^i (k_{b_k}^2 - z_k^2)\}. \quad (25)$$

For any $2 \leq i \leq n$, α_{i-1} is expressed as

$$\alpha_{i-1} = \frac{1}{g_{i-1}}(-f_{i-1} + \dot{\alpha}_{i-2} - (k_{b_{i-1}}^2 - z_{i-1}^2)k_{i-1}z_{i-1} - \frac{k_{b_{i-1}}^2 - z_{i-1}^2}{k_{b_{i-2}}^2 - z_{i-2}^2} g_{i-2} z_{i-2}, \quad (26)$$

where g_{i-1} and $-f_{i-1}$ is the function of $\bar{x}_{i-1}$ and $\dot{\alpha}_{i-2}$ is the function of $\bar{x}_{i-1}$ and $\bar{y}_d^{i-1}$. Since $\bar{x}_{i-1}$ is the function of $\bar{z}_{i-1}$ and $\bar{\alpha}_{i-2}$, we can represent α_{i-1} using only $\bar{z}_{i-1}$, $\bar{y}_d^{i-1}$, and $\bar{\alpha}_{i-2}$. Repeating the same process for $\bar{\alpha}_{i-2}$ and on, we eventually obtain α_1. Because the remaining α_1 is obviously the function of the z_1 and $y_d^{(1)}$, so we can express α_{i-1} using $\bar{z}_{i-1}$ and $\bar{y}_d^{i-1}$ for $2 \leq i \leq n$.

From the above result, $x_i = z_i + \alpha_{i-1}$ is the function of $\bar{z}_i$ and $\bar{y}_d^{i-1}$ which are respectively in the closed set $\Omega_{\bar{z}_i}$ and $\Omega_{\bar{y}_d^{i-1}}$, and hence x_i is bounded and the $\max|x_i|$ exists. Since the $\max|x_i| < k_{c_i}$ for $k_{c_i} > 0$, (i) follows.
(ii) From (25), we have $\Omega_{\bar{z}_n} = \{\bar{z}_n \in R^n | \Pi_{k=1}^n (k_{b_k}^2 - z_k^2(0)) \leq \Pi_{k=1}^n (k_{b_k}^2 - z_k^2)\}$. Obviously, the error state z_i, $i = 1, ..., n$, asymptotically goes to zero, due to $\dot{V} < 0$ in $\Omega_{\bar{z}_n} - \{0\}$. ■

One of the consequences in theorem 1 is that the control parameters region becomes larger than that in [12], which can be explained in the following remark.

Remark 1. In [12], $X_i(\xi)$ is obtained as the $k_{b_i} + \max|\alpha_{i-1}|$ in $\bar{\Omega}_{\bar{z}_{i-1}} = \{\bar{z}_{i-1} \in R^n : |z_j| \leq D_{z_j} = k_{b_j} \cdot \sqrt{1 - \Pi_{k=1}^n (k_{b_k}^2 - z_k^2(0))/\Pi_{k=1}^n k_{b_k}^2}, j = 1, ..., i-1\}, i = 2, ..., n$. However, in this paper, we obtain it as $\max|z_i + \alpha_{i-1}|$ in $\Omega_{\bar{z}_i}$. Since $D_{z_i} < k_{b_i}$ and $\Omega_{\bar{z}_i} \subset \bar{\Omega}_{\bar{z}_i}$, $i = 1, ..., n$, as shown in Figure 1, it follows that

$$\max_{\bar{z}_i \in \Omega_{\bar{z}_i}} |z_i + \alpha_{i-1}| \leq \max_{z_i \leq D_{z_i}} |z_i| + \max_{\bar{z}_{i-1} \in \Omega_{\bar{z}_{i-1}}} |\alpha_{i-1}| \leq D_{z_i} + \max_{\bar{z}_{i-1} \in \bar{\Omega}_{\bar{z}_{i-1}}} |\alpha_{i-1}| < k_{b_i} + \max_{\bar{z}_{i-1} \in \bar{\Omega}_{\bar{z}_{i-1}}} |\alpha_{i-1}|. \quad (27)$$

Under the same control parameters $\xi = [k_1, ..., k_{i-1}, k_{b_2}, ..., k_{b_i}]^T$, $\max|z_i + \alpha_{i-1}|$ in $\Omega_{\bar{z}_i}$ is smaller than $k_{b_i} + \max|\alpha_{i-1}|$ in $\bar{\Omega}_{\bar{z}_{i-1}}$. And it becomes larger up to $k_{b_i} + \max|\alpha_{i-1}|$ as k_{b_i} become larger. And then, given k_{c_i}, the region k_{b_i} satisfying condition (17) becomes larger than that in [12]. In addition, the region of initial conditions satisfying (18) becomes larger, and the region of attraction can also be extended.

III. Bounded Controller Using Backstepping Technique with Bounded Lyapunov Function

Refering to the nonlinear system with strictly feedback form, the control Lyapunov function V can be obtained. Let $\dot{V} = a(x) + b(x)u$, and using the result in [15], consider the

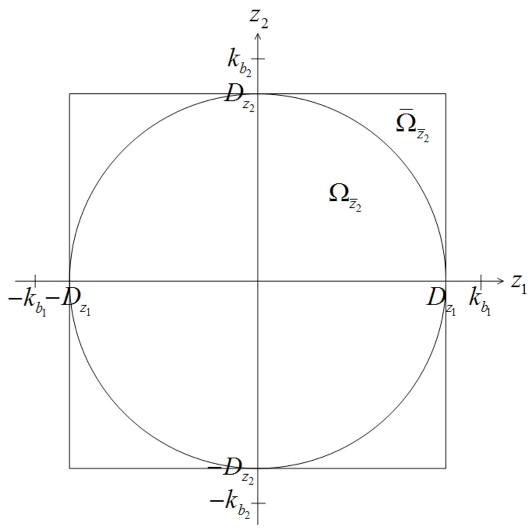

Fig. 1. The interconnection between k_{b_1}, k_{b_2}, D_{z_1}, D_{z_2}, $\Omega_{\bar{z}_2}$, and $\bar{\Omega}_{\bar{z}_2}$.

bounded state feedback control law:

$$u = \begin{cases} -\dfrac{a(x) + \sqrt{a(x)^2 + (\bar{u} \cdot b(x))^4}}{b(x)[1 + \sqrt{1 + (\bar{u} \cdot b(x))^2}]}, & b(x) \neq 0 \\ 0, & b(x) = 0 \end{cases} \tag{28}$$

where $\bar{u}$ is the maximum value of $|u|$. For this controller, if the state trajectory evolves within the state space region described by the set

$$\Phi(\bar{u}) = \{x \in R^n : a(x) < \bar{u}|b(x)|\}, \tag{29}$$

then it enforces $\dot{V} < 0$ while satisfying the input bounds.

As we have developed in the previous section, we now propose to use the barrier Lyapunov function V in (10) as a control Lyapunov function. Using the backstepping technique, we obtain $\dot{V} = a(z) + b(z)u$ where

$$a(z) = -\sum_{i=1}^{n-1} k_i z_i^2 + \frac{g_{n-1} z_{n-1} z_n}{k_{b_{n-1}}^2 - z_{n-1}^2} + \frac{(\dot{\alpha}_{n-1} - f_n) z_n}{k_{b_n}^2 - z_n^2}, \tag{30}$$

$$b(z) = \frac{g_n z_n}{k_{b_n}^2 - z_n^2}, \tag{31}$$

and we can always find u that make $\dot{V} < 0$. $\dot{V} < 0$ is guaranteed even if $b(z) = 0$, because $b(z) = 0$ means $z_n = 0$, which means $a(z) = -\sum_{i=1}^{n-1} k_i z_i^2 < 0$ in $\Omega_{\bar{z}_{n-1}} - \{0\}$. Using this CLF and the above bounded controller, we stabilize the system (1) while satisfying the state and input bounds as follows.

Under the same barrier Lyapunov function V as in (10), and the same α_i, $i = 1, ..., n-1$, as in (11) and (12), we construct the new input instead of (13) using the bounded control law

$$u = \begin{cases} -\dfrac{a(z) + \sqrt{a(z)^2 + (\bar{u} \cdot b(z))^4}}{b(z)[1 + \sqrt{1 + (\bar{u} \cdot b(z))^2}]}, & b(z) \neq 0 \\ 0, & b(z) = 0 \end{cases} \tag{32}$$

The time derivative of V in (10) can be rewritten as

$$\dot{V} = \frac{-\sqrt{a(z)^2 + (\bar{u} \cdot b(z))^4} + a(z)\sqrt{1 + (\bar{u} \cdot b(z))^2}}{1 + \sqrt{1 + (\bar{u} \cdot b(z))^2}}. \tag{33}$$

And then, $\dot{V} < 0$ whenever the condition $a(z) < \bar{u}|b(z)|$ for all $\bar{z}_n \in \Omega_{\bar{z}_n} - \{0\}$ is satisfied.

Theorem 2. Consider the closed loop system (15). Given $k_{c_i} > 0$, $i = 2, ..., n$, if we design the control parameters ξ such that the following conditions are satisfied,,

$$C1) \quad X_i(\xi) < k_{c_i}, \quad i = 2, ..., n \tag{34}$$

$$C2) \quad |z_i(0)| < k_{b_i}, \quad i = 1, ..., n \tag{35}$$

$$C3) \quad \Phi(\bar{u}) = \{\bar{z}_n \in R^n : a(z) < \bar{u}|b(z)|\}, \quad \forall \bar{z}_n \in \Omega_{\bar{z}_n} - \{0\} \tag{36}$$

then, the controller designed with the above design parameters ξ makes the closed loop system that satisfies:
(i) every state $x_i(t)$ remains in the compact set $x_i \leq X_i < k_{c_i}$, $i = 1, ..., n$.
(ii) every error state $z_i(t)$, $i = 1, ..., n$, converges to the origin within the compact set $\Omega_{\bar{z}_n}$.
(iii) the input bound is satisfied, that is, $|u| < \bar{u}$.

Proof: The proofs (i), (ii) are the same as those of Theorem 1. From the third condition (36), if $b(z) > 0$, we have

$$\begin{aligned} |u| &= \frac{a(z) + \sqrt{a(z)^2 + (\bar{u} \cdot b(z))^4}}{b(z)[1 + \sqrt{1 + (\bar{u} \cdot b(z))^2}]} \\ &< \frac{\bar{u} \cdot b(z) + \sqrt{(\bar{u} \cdot b(z))^2 + (\bar{u} \cdot b(z))^4}}{b(z)[1 + \sqrt{1 + (\bar{u} \cdot b(z))^2}]} \\ &= \frac{\bar{u} \cdot b(z) + \bar{u} \cdot b(z)\sqrt{1 + (\bar{u} \cdot b(z))^2}}{b(z)[1 + \sqrt{1 + (\bar{u} \cdot b(z))^2}]} = \bar{u} \end{aligned} \tag{37}$$

If $b(z) < 0$, we have

$$\begin{aligned} |u| &= -\frac{a(z) + \sqrt{a(z)^2 + (\bar{u} \cdot b(z))^4}}{b(z)[1 + \sqrt{1 + (\bar{u} \cdot b(z))^2}]} \\ &< -\left[\frac{-\bar{u} \cdot b(z) + \sqrt{(\bar{u} \cdot b(z))^2 + (\bar{u} \cdot b(z))^4}}{b(z)[1 + \sqrt{1 + (\bar{u} \cdot b(z))^2}]}\right] \\ &= -\left[\frac{-\bar{u} \cdot b(z) - \bar{u} \cdot b(z)\sqrt{1 + (\bar{u} \cdot b(z))^2}}{b(z)[1 + \sqrt{1 + (\bar{u} \cdot b(z))^2}]}\right] = \bar{u} \end{aligned} \tag{38}$$

and (iii) follows. Of course, we need to check this condition only for all $\bar{z}_n$ in $\Omega_{\bar{z}_n} - \{0\}$ to satisfy the state bounds with input bound. ■

Remark 2. The control Lyapunov function needs to be continuously differentiable, proper, and positive definite from definition 2. However, barrier Lyapunov function V in (10) regarded as the control Lyapunov function is not proper in the strict sense, because $V(z)$ is not defined as $|z_i| \to \infty$. But, in this paper, we do not consider the global stabilization but only the local stabilization in the domain of $V(z)$. So, we can regard this V as control Lyapunov function only if $V(z) \to \infty$ as $|z_i| \to k_{b_i}$ is satisfied for all $i = 1, ..., n$.

Remark 3. Many solutions satisfying the three conditions (34)-(36) can exist or not. If they exist, we prefer to choose one solution which maximize $J = \sum_{i=1}^{n-1} k_i + \sum_{i=2}^{n} k_{b_i}$. This objective function is designed since the k_i, $i = 1, ..., n-1$, and k_{b_i}, $i = 2, ..., n$, become larger, convergence rate becomes faster and the region of initial conditions satisfying (18) becomes larger.

IV. Simulation Result

Consider the second-order nonlinear system,

$$\begin{aligned} \dot{x}_1 &= 0.1x_1^2 + x_2, \\ \dot{x}_2 &= 0.1x_1x_2 - 0.2x_1 + (1 + x_1^2)u. \end{aligned} \tag{39}$$

We consider the problem subject to state bounds $|x_1| < k_{c_1} = 0.8$, $|x_2| < k_{c_2} = 2.5$, and input bound constraint $|u| < \bar{u} = 10$. For simplicity, we consider the desired trajectory $y_d = 0$, and we set initial conditions $x_1(0) = -0.45$ and $x_2(0) = -1.75$ and denote $\xi = [k_1, k_{b_2}]^T$.

Step 1. Since $y_d = 0$, $k_{b_1} = k_{c_1} - Y_1 = 0.8$, and the initial condition $z_1(0) < k_{b_1}$ is satisfied.

Step 2. Find a solution $\xi^* = [k_1^*, k_{b_2}^*]^T$ of the static optimization problem as follows:

$$\max_{k_1, k_{b_2} > 0} J := k_1 + k_{b_2}, \tag{40}$$

subject to the constraints:

$$\begin{aligned} X_2(\xi) &< k_{c_2}, & (41) \\ |z_2(0)| &< k_{b_2}, & (42) \\ a(z) &< \bar{u}|b(z)|, \quad \forall \bar{z}_2 \in \Omega_{\bar{z}_2} - \{0\} & (43) \end{aligned}$$

where

$$X_2(\xi) = \max_{\bar{z}_2 \in \Omega_{\bar{z}_2}, \bar{y}_d^1 \in \Omega_{y_d^1}} |z_2 - 0.1z_1^2 - (0.8^2 - z_1^2)k_1 z_1|, \tag{44}$$

$$z_2(0) = -1.75 - (-0.0203 + 0.1969k_1), \tag{45}$$

$$\begin{aligned} \Omega_{\bar{z}_2} = \{ \bar{z}_2 \in R^2 : &\, 0.4375(k_{b_2}^2 - z_2(0)^2) \\ &\leq (0.8^2 - z_1^2)(k_{b_2}^2 - z_2^2) \}. \end{aligned} \tag{46}$$

We obtain $k_1^* = 1.1234$ and $k_{b_2}^* = 1.9530$. In solving this optimization problem, we used the built-in functions, fmincon and fseminf in Matlab.

Step 3. Implement the control input u using the bounded control law (32).

Remark 4. Referring to Remark 1, we compare the regions of attraction via this example when we obtain $X_2(\xi)$ as $k_{b_2} + \max|\alpha_1|$ in $\bar{\Omega}_{z_1}$ and as $\max|z_2 + \alpha_1|$ in $\Omega_{\bar{z}_2}$. Since we used the backstepping controller based on barrier Lyapunov function in remark 1, we removed condition (43) and used the control input u (9) in the above example. Under these conditions, we drew the regions of attraction when we obtain $X_2(\xi)$ as $k_{b_2} + \max|\alpha_1|$ in Figure 2 and as $\max|z_2 + \alpha_1|$ in Figure 3. The region of attraction in Figure 3 is larger than that in

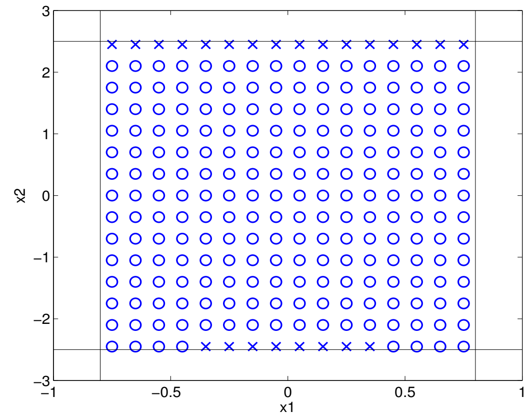

Fig. 2. The region of attraction of the backstepping controller based on barrier Lyapunov function when we obtain $X_2(\xi)$ as $k_{b_2} + \max|\alpha_1|$ in $\bar{\Omega}_{z_1}$ as in [12].

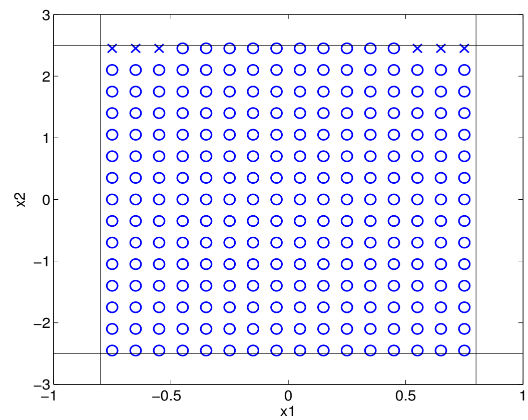

Fig. 3. The region of attraction of the backstepping controller based on barrier Lyapunov function when we obtain $X_2(\xi)$ as $\max|z_2 + \alpha_1|$ in $\Omega_{\bar{z}_2}$.

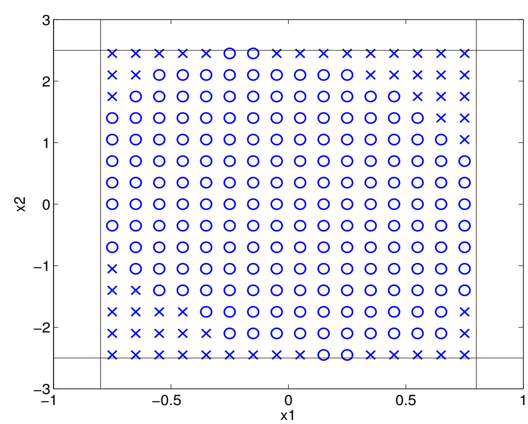

Fig. 4. The region of attraction of the proposed bounded controller.

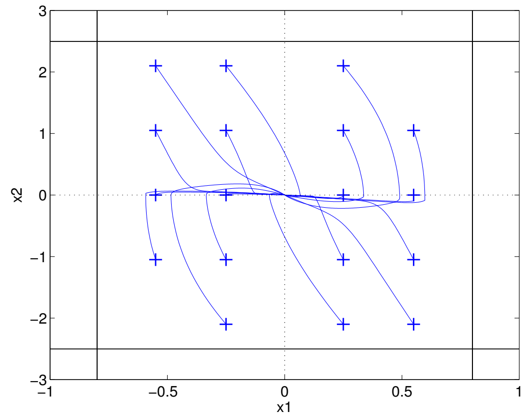

Fig. 5. The state trajectories when the proposed bounded controller is used.

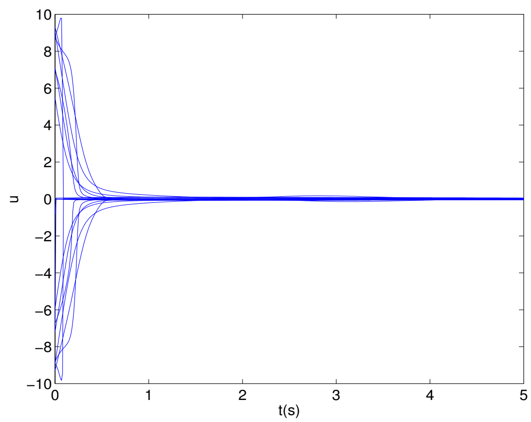

Fig. 6. The input trajectories when the proposed bounded controller is used. Each of the control inputs is created corresponding to the initial conditions given in Figure 5.

Figure 2. Therefore, with proposed bounded controller, this system can be made stable even if states start from the larger region.

Remark 5. Coming back to the bounded controller, we need to check whether this controller can make the closed loop system safisfy the input bounds or not. With the condition (43) and using control input u (32) instead of (9), we drew the region of attraction in Figure 4. As shown in Figure 4, the region of attraction shrinks because of additional condition (43). However, the input with bounded controller is bounded by $\bar{u}$ as shown in Figure 6, and the states are also bounded by state constraints k_{c_1}, k_{c_2} as shown in Figure 5.

V. Conclusion

In this paper, we consider a class of nonlinear systems with state and input bounds and propose a bounded controller using the backstepping technique with barrier Lyapunov function. In contrast to the conventional controller that ensures only input boundness and satisfies the state bounds, the proposed controller guarantees stability of the closed loop nonlinear systems in strict feedback form that satisfies the given state and input bounds. The simulation results show that the proposed controller efficiently controls the nonlinear system in strict feedback form while satisfying the state as well as input bounds.

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