Asymptotic Stability Analysis of TSK Fuzzy PI Control System Using Circle Criterion

Jongkyoo Kim, Ok-Kyu Choi,
Hyosin Kim
Department of Electrical Engineering
Pohang University of Science and Technology (POSTECH)
Pohang, Gyeongbuk 790 784, Republic of Korea
Email: jong9@postech.ac.kr, skaruwls@postech.ac.kr,
foruever@postech.ac.kr

Jin S. Lee
Departments of Creative IT Engineering
and Electrical Engineering
Pohang University of Science and Technology (POSTECH)
Pohang, Gyeongbuk 790 784, Republic of Korea
Email: jsoo@postech.ac.kr

***Abstract*—In this paper, we study asymptotic stability of the Takagi-Sugeno-Kang (TSK) fuzzy PI control system using the circle criterion theorem. We first divide a TSK fuzzy PI controller into a memoryless nonlinear part and a linear dynamic unit, and merge the linear dynamic unit with a linear plant to be controlled and form a new augmented plant. The system is then reconstructed as a new interconnected form of the augmented linear plant and the memoryless nonlinear part. We then derive transfer function matrix of the augmented linear plant and sector condition of the memoryless nonlinear part. Based on the derived transfer function matrix and the sector condition, we determine sufficient conditions that guarantee asymptotic stability of the TSK fuzzy PI control system by using the circle criterion. Two examples are presented to demonstrate the feasibility of the proposed method.**

I. INTRODUCTION

Fuzzy controllers can be designed to accommodate the plant characteristics acquired from the operators' experience with the plant. They have thus been applied to many plants the mathematical modeling of which is impossible or difficult to derive but which the human experts generally understand its overall characteristics from experience. The first fuzzy controller was introduced by Mamdani and Assilian [1], [2], using the fuzzy set theory developed by Zadeh [3]. Another type of fuzzy controllers was later proposed by Takagi, Sugeno, and Kang [4], [5].

Many fuzzy control systems have been analyzed in the past two decades. One approach is to analyze the stability of fuzzy controllers with Takagi-Sugeno fuzzy models [6], [7] and another approach is to analyze the stability of fuzzy controllers with mathematical models. In the second approach, more results are the bounded-input bounded-output stability ones by using the small gain theorem [8]–[15] and less results are the asymptotic stability ones by using frequency-domain method [16]–[22].

The circle criterion is a well known frequency-domain method that gives sufficient conditions for asymptotic stability of the closed loop system. The system with P type TSK fuzzy controller was analyzed by using the circle criterion [21]. However, in many applications, TSK fuzzy PI controller is more general and useful than TSK fuzzy P controller because the integral control action provides important practical advantage for output regulation. Thus, TSK fuzzy PI control system was analyzed using the circle criterion in [22]. They reduced the problem to the scalar case under some conditions on the input fuzzy sets, the membership functions and the design parameters and considered it as single-input single-output case. Thus, the sufficient condition with graphical interpretation was derived but its applicability was restricted to single-input plant case.

In this paper, we study asymptotic stability of the TSK fuzzy PI control system using the circle criterion which can be deployed to the multi-input plant case. To apply the circle criterion, we impose two symmetry constraints on the used input membership functions and on the design parameters of the TSK fuzzy PI controller. We divide the controller into a memoryless nonlinear part and a linear dynamic unit. By merging the linear dynamic unit with the linear plant under consideration, we arrange the system as a feedback loop of an augmented linear plant and a memoryless nonlinear part. Then we derive the transfer function matrix of the augmented linear plant and the sector condition of the memoryless nonlinear part. Based on the derived transfer function matrix and the sector condition, we determine the sufficient conditions that guarantee asymptotic stability of the TSK fuzzy PI control system by applying the circle criterion to the reconstructed system. This paper is organized as follows. In section II, TSK fuzzy PI control system is presented. In section III, the asymptotic stability of the TSK fuzzy PI control system is analyzed by using the circle criterion and sufficient conditions that guarantee closed system stability are determined. In section IV, an example is presented to demonstrate the feasibility of the sufficient conditions developed in section III. In section V, we draw conclusions.

II. TSK FUZZY PI CONTROL SYSTEM

The TSK fuzzy PI control system we consider in this study consists of a TSK fuzzy PI controller and the given plant as shown in Fig. 1, where $r(t)$ is the reference input, $u(t)$ is the fuzzy controller output, $y(t)$ is the plant output, and $e(t) = r(t) - y(t)$ denotes the error. The plant is a linear time invariant

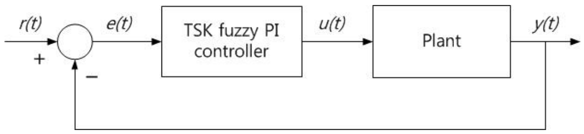


Fig. 1: Structure of the TSK fuzzy PI control system.

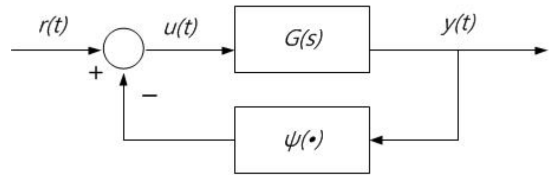


Fig. 2: Feedback connection which consists of linear dynamic system and sector-bounded nonlinearity .

(LTI) system which can be described as follows:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (1)$$

$$y(t) = Cx(t) + Du(t), \quad (2)$$

where $x(t) \in R^n$, $u(t), y(t) \in R$. The employed TSK fuzzy PI controller uses singleton fuzzifier, center average defuzzifier, and fuzzy logic AND operator of either Zadeh or Larsen, as commonly used in the majority of fuzzy controllers. The TSK PI controller can be computed directly or by using the derivative form. When using the derivative form, the derivative signal is first determined from the error and its derivative and the controller output is then obtained by integrating the derivative signal. In this paper, we employ the derivative form when computing the controller output.

Let N_z denote the number of fuzzy sets associated with the input variable z, where z is either $e(t)$ or $\dot{e}(t)$. Each input fuzzy set is then denoted as $A_z^{l_z}$ and characterized by input membership functions $\mu_z^{l_z}$, where $1 \leq l_z \leq N_z$ is an index for input fuzzy sets.

The rules of the TSK fuzzy PI controller consist of the premise parts that are formed by using the membership functions of the input fuzzy sets and the consequent parts that are simply the linear combinations of input values. Given N number of rules, the ith rule R_i for $1 \leq i \leq N$ is defined as:

$$\begin{aligned} R_i = \ & \text{IF } e(t) \text{ is } A_e^{l_e} \text{ AND } \dot{e}(t) \text{ is } A_{\dot{e}}^{l_{\dot{e}}} \\ & \text{THEN } \dot{u}_i(t) \text{ is } \alpha_i \cdot e(t) + \beta_i \cdot \dot{e}(t), \end{aligned} \quad (3)$$

where $\dot{u}_i(t)$ is the contribution of the ith rule to the derivative signal $\dot{u}(t)$ and α_i and β_i are the design parameters. Because we use two inputs $(e(t), \dot{e}(t))$, the total number of rules N is equal to $N_e \cdot N_{\dot{e}}$. In each rule, the relation between the fuzzy rule index i and the fuzzy set indices l_e and $l_{\dot{e}}$ can be expressed as:

$$i = (l_e - 1) \cdot N_{\dot{e}} + l_{\dot{e}}. \quad (4)$$

The rules constructed in this manner cover all possible combinations of input fuzzy sets. The derivative signal of the TSK fuzzy PI controller can then be formulated as:

$$\begin{aligned} \dot{u}(t) =& \frac{\sum_{i=1}^{N} \dot{u}_i(t) \cdot \mu_i(e,\dot{e})}{\sum_{i=1}^{N} \mu_i(e,\dot{e})} \\ =& \frac{\sum_{i=1}^{N} (\alpha_i \cdot e(t) + \beta_i \cdot \dot{e}(t)) \cdot \mu_i(e,\dot{e})}{\sum_{i=1}^{N} \mu_i(e,\dot{e})} \\ =& \sum_{i=1}^{N} (\alpha_i \cdot e(t) + \beta_i \cdot \dot{e}(t)) \cdot \bar{\mu}_i(e,\dot{e}) \\ =& \sum_{i=1}^{N} \bar{\mu}_i(e,\dot{e}) \cdot \alpha_i \cdot e(t) + \sum_{i=1}^{N} \bar{\mu}_i(e,\dot{e}) \cdot \beta_i \cdot \dot{e}(t) \\ =& K_I(e,\dot{e}) \cdot e(t) + K_P(e,\dot{e}) \cdot \dot{e}(t), \end{aligned} \quad (5)$$

where μ_i is the compatibility function of the ith rule and $\bar{\mu}_i$ is the normalized μ_i as follows:

$$\begin{aligned} \mu_i(e,\dot{e}) =& \text{MIN}(\mu_e^{l_e}(e(t)), \mu_{\dot{e}}^{l_{\dot{e}}}(\dot{e}(t))) \\ & \text{or } \mu_e^{l_e}(e(t)) \cdot \mu_{\dot{e}}^{l_{\dot{e}}}(\dot{e}(t)), \end{aligned} \quad (6)$$

$$\bar{\mu}_i(e,\dot{e}) = \frac{\mu_i(e,\dot{e})}{\sum_{j=1}^{N} \mu_j(e,\dot{e})}, \quad (7)$$

and the control gains $K_P(e,\dot{e})$ and $K_I(e,\dot{e})$ are defined as:

$$K_P(e,\dot{e}) = \sum_{i=1}^{N} \bar{\mu}_i(e,\dot{e}) \cdot \beta_i, \quad (8)$$

$$K_I(e,\dot{e}) = \sum_{i=1}^{N} \bar{\mu}_i(e,\dot{e}) \cdot \alpha_i. \quad (9)$$

The control output is then obtained by integrating this derivative signal:

$$u(t) = \int_0^t \dot{u}(\tau) d\tau. \quad (10)$$

The input membership function $\mu_z^{l_z}$ and the design parameters α_i and β_i of the TSK fuzzy PI controller are assumed to be symmetric in the sense that:

$$\mu_z^{l_z}(z) = \mu_z^{N_z - l_z + 1}(-z), \text{ for all } 1 \leq l_z \leq N_z, \quad (11)$$

$$\alpha_i = \alpha_{N-i+1} \text{ and } \beta_i = \beta_{N-i+1}, \text{ for all } 1 \leq i \leq N. \quad (12)$$

Because α_i, β_i are design parameters, we can choose them in such a way that (12) is satisfied.

III. Stability analysis

In this section, we determine sufficient conditions under which the TSK fuzzy PI control system described in section II becomes absolutely stable. The system is said to absolutely stable if its equilibrium point is globally uniformly asymptotically stable for all nonlinearities in a given sector. The circle criterion method [23] gives frequency-domain sufficient conditions under which the closed loop system becomes absolutely stable.

Now, we will transform the TSK fuzzy PI control system into a feedback connection form (Fig. 2) and employ the circle criterion to analyze the stability of the closed loop system.

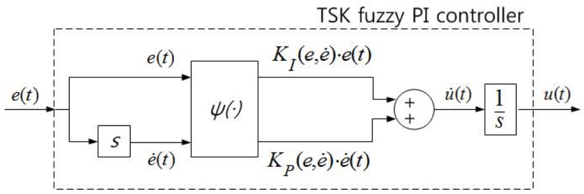


Fig. 3: Structure of the TSK fuzzy PI controller.

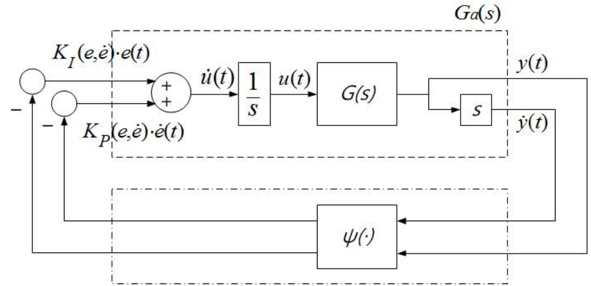


Fig. 4: Structure of the rearranged TSK fuzzy PI control system.

Examining the controller output (5), (10), we find that the TSK fuzzy PI controller can be decomposed into a two-input two-output memoryless nonlinear function, a differentiator at the input, summation and an integrator at the output. So the controller can be divided into a memoryless nonlinear part and a linear dynamic part (Fig. 3). The memoryless nonlinear part is defined as:

$$\psi\left(\begin{bmatrix} e(t) \\ \dot{e}(t) \end{bmatrix}\right) = \begin{bmatrix} K_I(e,\dot{e}) \cdot e(t) \\ K_P(e,\dot{e}) \cdot \dot{e}(t) \end{bmatrix}. \tag{13}$$

From the constraint (11) and the fuzzy rule index (4), the compatibility function (6) must satisfy

$$\begin{aligned} \mu_i(e,\dot{e}) =& \text{MIN}(\mu_e^{l_e}(e(t)), \mu_{\dot{e}}^{l_{\dot{e}}}(\dot{e}(t))) \\ & \text{ or } \mu_e^{l_e}(e(t)) \cdot \mu_{\dot{e}}^{l_{\dot{e}}}(\dot{e}(t)) \\ =& \text{MIN}(\mu_e^{N_e-l_e+1}(-e(t)), \mu_{\dot{e}}^{N_{\dot{e}}-l_{\dot{e}}+1}(-\dot{e}(t))) \\ & \text{ or } \mu_e^{N_e-l_e+1}(-e(t)) \cdot \mu_{\dot{e}}^{N_{\dot{e}}-l_{\dot{e}}+1}(-\dot{e}(t)) \\ =& \mu_{N-i+1}(-e,-\dot{e}). \end{aligned} \tag{14}$$

Using the constraint (12) and the compatibility function (14), it follows that

$$\begin{aligned} K_P(-e,-\dot{e}) &= \frac{\sum_{i=1}^N \beta_i \cdot \mu_i(-e,-\dot{e})}{\sum_{i=1}^N \mu_i(-e,-\dot{e})} \\ &= \frac{\sum_{i=1}^N \beta_{N-i+1} \cdot \mu_{N-i+1}(e,\dot{e})}{\sum_{i=1}^N \mu_{N-i+1}(e,\dot{e})} \\ &= \frac{\sum_{j=1}^N \beta_j \cdot \mu_j(e,\dot{e})}{\sum_{j=1}^N \mu_j(e,\dot{e})} \\ &= K_P(e,\dot{e}), \end{aligned} \tag{15}$$

and

$$K_I(-e,-\dot{e}) = K_I(e,\dot{e}) \tag{16}$$

following the same procedure. Consequently, the memoryless nonlinear part of the TSK fuzzy PI controller formed with the two constraints becomes an odd function:

$$\begin{aligned} \psi\left(\begin{bmatrix} -e(t) \\ -\dot{e}(t) \end{bmatrix}\right) &= \begin{bmatrix} K_I(-e,-\dot{e}) \cdot (-e(t)) \\ K_P(-e,-\dot{e}) \cdot (-\dot{e}(t)) \end{bmatrix} \\ &= -\begin{bmatrix} K_I(-e,-\dot{e}) \cdot e(t) \\ K_P(-e,-\dot{e}) \cdot \dot{e}(t) \end{bmatrix} \\ &= -\begin{bmatrix} K_I(e,\dot{e}) \cdot e(t) \\ K_P(e,\dot{e}) \cdot \dot{e}(t) \end{bmatrix} \\ &= -\psi\left(\begin{bmatrix} e(t) \\ \dot{e}(t) \end{bmatrix}\right). \end{aligned} \tag{17}$$

When $r(t) = 0$, we have

$$\begin{aligned} \psi\left(\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}\right) &= \psi\left(\begin{bmatrix} -e(t) \\ -\dot{e}(t) \end{bmatrix}\right) \\ &= -\begin{bmatrix} K_I(e,\dot{e}) \cdot e(t) \\ K_P(e,\dot{e}) \cdot \dot{e}(t) \end{bmatrix}, \end{aligned} \tag{18}$$

and the TSK fuzzy PI control system can be rearranged as in Fig. 4. The augmented LTI system $G_a(s)$ (Fig. 4) is the combination of the linear dynamic part of the controller and the plant under consideration. This augmented system is then rearranged as:

$$\dot{x}_a(t) = A_a x_a(t) + B_a u_a(t), \tag{19}$$

$$y_a(t) = C_a x_a(t) + D_a u_a(t), \tag{20}$$

where $x_a(t) = [x(t)\ u(t)]^T \in R^{n+1}$, $u_a(t) = [K_I(e,\dot{e}) \cdot e(t)\ K_P(e,\dot{e}) \cdot \dot{e}(t)]^T \in R^2$, $y_a(t) = [y(t)\ \dot{y}(t)]^T \in R^2$,

$$A_a = \begin{pmatrix} A_{n\times n} & B_{n\times 1} \\ 0_{1\times n} & 0_{1\times 1} \end{pmatrix}, \tag{21}$$

$$B_a = \begin{pmatrix} 0_{n\times 1} & 0_{n\times 1} \\ I_{1\times 1} & I_{1\times 1} \end{pmatrix}, \tag{22}$$

$$C_a = \begin{pmatrix} C_{1\times n} & D_{1\times 1} \\ C_{1\times n}A_{n\times n} & C_{1\times n}B_{n\times 1} \end{pmatrix}, \tag{23}$$

$$D_a = \begin{pmatrix} 0_{1\times 1} & 0_{1\times 1} \\ D_{1\times 1} & D_{1\times 1} \end{pmatrix}. \tag{24}$$

The transfer function matrix of the augmented system is then

$$G_a(s) = C_a(sI - A_a)^{-1}B_a + D_a. \tag{25}$$

Next, we will find the sector condition of ψ. Using (8) and (9), the memoryless nonlinear function (13) can be represented as:

$$\begin{aligned} \psi\left(\begin{bmatrix} e(t) \\ \dot{e}(t) \end{bmatrix}\right) &= \begin{bmatrix} \sum_{i=1}^N \bar{\mu}_i(e,\dot{e}) \cdot \alpha_i \cdot e(t) \\ \sum_{i=1}^N \bar{\mu}_i(e,\dot{e}) \cdot \beta_i \cdot \dot{e}(t) \end{bmatrix} \\ &= \begin{bmatrix} (\bar{\mu}_1(e,\dot{e}) \cdot \alpha_1 + \bar{\mu}_2(e,\dot{e}) \cdot \alpha_2 \\ (\bar{\mu}_1(e,\dot{e}) \cdot \beta_1 + \bar{\mu}_2(e,\dot{e}) \cdot \beta_2 \\ \quad + \cdots + \bar{\mu}_N(e,\dot{e}) \cdot \alpha_N) \cdot e(t) \\ \quad + \cdots + \bar{\mu}_N(e,\dot{e}) \cdot \beta_N) \cdot \dot{e}(t) \end{bmatrix}. \end{aligned} \tag{26}$$

Since the normalized compatibility function must satisfy

$$0 \le \bar{\mu}_i(e(t), \dot{e}(t)) \le 1, \text{ for all } 1 \le i \le N, \quad (27)$$

$$\sum_{i=1}^{N} \bar{\mu}_i(e(t), \dot{e}(t)) = 1, \quad (28)$$

we have

$$\min_i(\alpha_i) \le \bar{\mu}_1(e, \dot{e}) \cdot \alpha_1 + \bar{\mu}_2(e, \dot{e}) \cdot \alpha_2 + \cdots + \bar{\mu}_N(e, \dot{e}) \cdot \alpha_N \le \max_i(\alpha_i), \quad (29)$$

$$\min_i(\beta_i) \le \bar{\mu}_1(e, \dot{e}) \cdot \beta_1 + \bar{\mu}_2(e, \dot{e}) \cdot \beta_2 + \cdots + \bar{\mu}_N(e, \dot{e}) \cdot \beta_N \le \max_i(\beta_i). \quad (30)$$

Therefore, ψ belongs to the sector:

$$\psi\left(\begin{bmatrix} e(t) \\ \dot{e}(t) \end{bmatrix}\right) \in \left[\begin{pmatrix} \min_i(\alpha_i) & 0 \\ 0 & \min_i(\beta_i) \end{pmatrix}, \begin{pmatrix} \max_i(\alpha_i) & 0 \\ 0 & \max_i(\beta_i) \end{pmatrix}\right]. \quad (31)$$

We now apply the circle criterion to this rearranged system of Fig. 4 and derive the following theorem.

Theorem: The TSK fuzzy PI control system described in section II is absolutely stable if

- (A_a, B_a) is controllable and (A_a, C_a) is observable,
- $\psi \in [K_1, K_2]$, with $K = K_2 - K_1 = K^T$, and
- $I + KG_a(s)[I + K_1 G_a(s)]^{-1}$ is strictly positive real (SPR),

where $K_1 = \text{diag}(\min_i(\alpha_i), \min_i(\beta_i))$ and $K_2 = \text{diag}$ $(\max_i(\alpha_i), \max_i(\beta_i))$.

Checking SPR condition of the transfer function matrix is in general difficult and hence we adopt the linear matrix inequality (LMI) technique to check the third condition in the above theorem [20] , [24]. Since $G_a(s)[I+K_1G_a(s)]^{-1}$ corresponds to a plant $G_a(s)$ with a feedback gain K_1, we treat these interconnected units as one system and convert the transfer function matrix $I+KG_a(s)[I+K_1G_a(s)]^{-1}$ to an equivalent transfer function matrix $G_l(s) = C_l(sI - A_l)^{-1}B_l + D_l$ [25], where

$$A_l = A_a - B_a(I + K_1D_a)^{-1}K_1C_a, \quad (32)$$

$$B_l = B_a(I + K_1D_a)^{-1}, \quad (33)$$

$$C_l = K(I + D_aK_1)^{-1}C_a, \quad (34)$$

$$D_l = I + K(I + D_aK_1)^{-1}D_a. \quad (35)$$

Then the transfer function matrix $I+KG_a(s)[I+K_1G_a(s)]^{-1}$ is SPR if there exists a matrix $P = P^T \in R^{(n+1)\times(n+1)}$ such that the following LMI holds:

$$\begin{bmatrix} -A_l^T P - PA_l & C_l^T - PB_l & 0 \\ C_l - B_l^T P & D_l + D_l^T & 0 \\ 0 & 0 & P \end{bmatrix} > 0. \quad (36)$$

Remark 1. The extended circle criterion [20] was used to analyze the stability of TSK fuzzy PI control system in [22]. To reduce the problem to the scalar case, they used the same fuzzy sets and the same membership functions for two inputs e and $\dot{e}$. On the other hand, we can design more various TSK fuzzy PI controller using different number of fuzzy sets and different shapes of membership functions for each input e and $\dot{e}$. Also, the extended circle criterion has difficulty in employing it to the multi-input multi-output (MIMO) case but our method can be easily extended to the MIMO case because the multivariable circle criterion is used in this paper.

Remark 2. For the TSK fuzzy PD control system, we can also divide the system into a memoryless nonlinear unit and a linear dynamic unit. Instead of using derivative form, we can directly compute the controller output and derive sufficient conditions by following the same procedure as used in the TSK fuzzy PI control system case. We can also extend this idea to the TSK fuzzy PID control system and derive the corresponding sufficient conditions in a similar manner.

IV. Numerical example

In this section, we give two examples to demonstrate how to use the above theorem in analyzing the stability of the TSK fuzzy PI control system.

Example 1. Consider a unstable plant

$$G(s) = \frac{0.1s + 1}{s^2 - s + 1}, \quad (37)$$

and one of its state space representation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad (38)$$

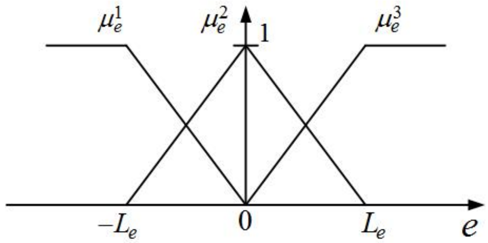
$$y(t) = \begin{bmatrix} 0.1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \quad (39)$$

First, we design a TSK fuzzy PI controller that uses three fuzzy sets for e and two fuzzy sets for $\dot{e}$. The fuzzy sets for e and for $\dot{e}$ are defined by using the triangular membership functions and the trapezoidal membership functions respectively which satisfy constraint (11) as given in Fig. 5. The number of rules is 6, and Zadeh AND operator is used. Control parameters were selected by trial and error so that they satisfy constraint (12). The selected parameters are $L_e = 1$, $L_{\dot{e}} = 2.5$, $\alpha_1 = \alpha_6 = 20$, $\alpha_2 = \alpha_5 = 30$, $\alpha_3 = \alpha_4 = 60$, $\beta_1 = \beta_6 = 8$, $\beta_2 = \beta_5 = 10$ and $\beta_3 = \beta_4 = 20$.

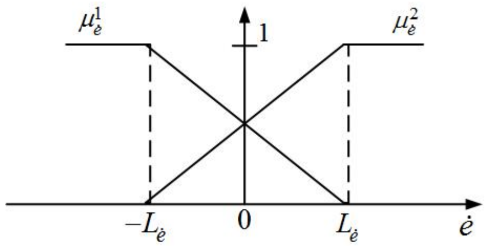
Now, we will check three sufficient conditions under which this TSK fuzzy PI control system becomes asymptotically stable. First, the augmented system $G_a(s) = C_a(sI-A_a)^{-1}+ B_a + D_a$ is represented as:

$$A_a = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad (40)$$

$$B_a = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad (41)$$



(a) Input membership functions for e.



(b) Input membership functions for $\dot{e}$.

Fig. 5: Input membership functions.

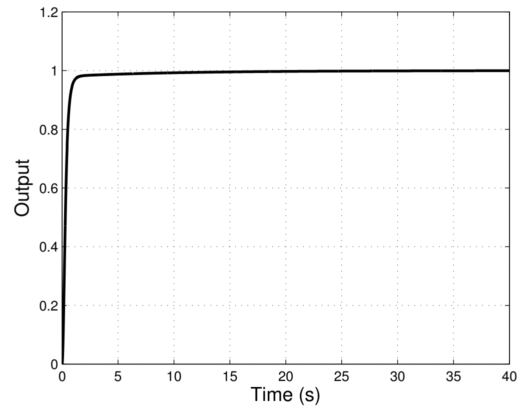


Fig. 6: Unit step response of the unstable plant with the TSK fuzzy PI controller.

$$C_a = \begin{pmatrix} 0.1 & 1 & 0 \\ -1 & 1.1 & 1 \end{pmatrix}, \tag{42}$$

$$D_a = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{43}$$

Note that (A_a, B_a) is controllable and (A_a, C_a) is observable. Next, the sector bounds are determined as $K_1 = \text{diag}(20, 8)$ and $K_2 = \text{diag}(60, 20)$ from the design parameters α_i and β_i. Finally, we need to check if $I + KG_a(s)[I + K_1G_a(s)]^{-1}$ is SPR. We now use LMI to check the SPR condition of the above transfer function matrix. To this end, we rewrite the transfer function matrix $I + KG_a(s)[I + K_1G_a(s)]^{-1}$ as $C_l(s) = (sI - A_l)^{-1}B_l + D_l$, where

$$A_l = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ 6 & -28.8 & -2 \end{pmatrix}, \tag{44}$$

$$B_l = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}, \tag{45}$$

$$C_l = \begin{pmatrix} 4 & 40 & 0 \\ -12 & 13.2 & 12 \end{pmatrix}, \tag{46}$$

$$D_l = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{47}$$

Then, using MATLAB, we were able to find

$$P = P^T = \begin{pmatrix} 91.9942 & 15.104 & -9.2569 \\ 15.104 & 598.1892 & 36.2748 \\ -9.2569 & 36.2748 & 12.5734 \end{pmatrix} \tag{48}$$

that satisfies (36). Therefore, we conclude that the equilibrium point $[x_1(t)\ x_2(t)]^T = [0\ 0]^T$ of this TSK fuzzy PI control system is asymptotically stable. The unit step response of the system is shown in Fig. 6.

Example 2. Consider an automobile cruise control system [26]

$$\begin{bmatrix} \dot{v}(t) \\ \dot{f}(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{m}(-A_\rho v^2(t) + f(t)) \\ \frac{1}{\tau}(-f(t) + u(t)) \end{bmatrix}, \tag{49}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} v(t) \\ f(t) \end{bmatrix}, \tag{50}$$

where $v(t)$ is the vehicle's speed, $u(t)$ is the control input, $m = 1300$ kg is the mass of the vehicle, $A_\rho = 0.3$ Ns^2/m^2 is its aerodynamic drag, $f(t)$ is the driving/braking force, and $\tau = 0.2$ sec is the engine/brake time constant. In this example, we develop a TSK fuzzy PI controller that regulates $v(t)$ to a desired value 10 m/s. Because the equilibrium point $[v_{eq}(t)\ f_{eq}(t)\ u_{eq}(t)]^T$ is $[10\ -30\ -30]^T$ when $v(t) = 10$ m/s, we can rewrite the system as

$$\begin{bmatrix} \dot{\tilde{v}}(t) \\ \dot{\tilde{f}}(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{m}(-A_\rho(\tilde{v} + 10)^2(t) + (\tilde{f}(t) + 30)) \\ \frac{1}{\tau}(-(\tilde{f}(t) + 30) + (\tilde{u}(t) + 30)) \end{bmatrix}, \tag{51}$$

$$\tilde{y}(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{v}(t) \\ \tilde{f}(t) \end{bmatrix}, \tag{52}$$

where $\tilde{v}(t) = v(t) - 10$, $\tilde{f}(t) = f(t) - 30$ and $\tilde{u}(t) = u(t) - 30$. Now, we design the TSK fuzzy PI controller using three input fuzzy sets, triangular membership functions for each e and $\dot{e}$ as given in Fig. 5a, and Zadeh AND operator. Control parameters were selected by trial and error so that they satisfy constraint (12). The selected parameters are $L_e = 10$, $L_{\dot{e}} = 1$, $\alpha_1 = \alpha_9 = 30$, $\alpha_2 = \alpha_8 = 60$, $\alpha_3 = \alpha_7 = 40$, $\alpha_4 = \alpha_6 = 30$, $\alpha_5 = 25$, $\beta_1 = \beta_9 = 350$, $\beta_2 = \beta_8 = 200$, $\beta_3 = \beta_7 = 300$, $\beta_4 = \beta_6 = 350$ and $\beta_5 = 450$.

To apply the theorem, we linearize the system around the origin, $[\tilde{v}(t)\ \tilde{f}(t)]^T = [0\ 0]^T$, as follows:

$$\begin{bmatrix}\dot{\tilde{v}}(t)\\ \dot{\tilde{f}}(t)\end{bmatrix} = \begin{bmatrix}-0.0046 & 0.0008\\ 0 & -5\end{bmatrix}\begin{bmatrix}\tilde{v}(t)\\ \tilde{f}(t)\end{bmatrix} + \begin{bmatrix}0\\ 5\end{bmatrix} u(t), \quad (53)$$

$$\tilde{y}(t) = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\tilde{v}(t)\\ \tilde{f}(t)\end{bmatrix}. \quad (54)$$

The corresponding matrix A_a, B_a, C_a, D_a are as follows:

$$A_a = \begin{pmatrix}-0.0046 & 0.0008 & 0\\ 0 & -5 & 5\\ 0 & 0 & 0\end{pmatrix}, \quad (55)$$

$$B_a = \begin{pmatrix}0 & 0\\ 0 & 0\\ 1 & 1\end{pmatrix}, \quad (56)$$

$$C_a = \begin{pmatrix}1 & 0 & 0\\ -0.0046 & 0.0008 & 0\end{pmatrix}, \quad (57)$$

$$D_a = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}. \quad (58)$$

Note that (A_a, B_a) is controllable and (A_a, C_a) is observable. Next, the sector bounds are determined as $K_1 = \text{diag}(25, 200)$ and $K_2 = \text{diag}(60, 450)$ from the design parameters α_i and β_i. Then, the transfer function matrix $G_l(s) = C_l(sI - A_l)^{-1}B_l + D_l$ is represented as:

$$A_l = \begin{pmatrix}-0.0046 & 0.0008 & 0\\ 0 & -5 & 5\\ -24.0769 & -0.1538 & 0\end{pmatrix}, \quad (59)$$

$$B_l = \begin{pmatrix}0 & 0\\ 0 & 0\\ 1 & 1\end{pmatrix}, \quad (60)$$

$$C_l = \begin{pmatrix}35 & 0 & 0\\ -1.1538 & 0.1923 & 0\end{pmatrix}, \quad (61)$$

$$D_l = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \quad (62)$$

Finally, we can find

$$P = P^T = \begin{pmatrix}15343 & -3 & 50\\ -3 & 7 & -6\\ 50 & -6 & 7\end{pmatrix} \quad (63)$$

that satisfies (36) using MATLAB. Therefore, we conclude that this TSK fuzzy PI control system is asymptotically stable around the origin or around $[v(t)\ \ f(t)]^T = [10\ 30]^T$. The vehicle speed from 0 m/s to 10 m/s is shown in Fig. 7.

V. Conclusion

In this paper, we derive sufficient conditions that guarantee asymptotic stability of the TSK fuzzy PI control system. We first divide the TSK fuzzy PI controller into a memoryless nonlinearity part and a linear dynamic system unit, and merge the linear dynamic system unit with the given linear plant to form an augmented linear plant. Then the system becomes a new feedback form of augmented linear plant and memoryless nonlinear part. Next, we determine the transfer function matrix of the augmented linear plant and the sector condition of the memoryless nonlinear part. The circle criterion method is then employed to derive asymptotic stability conditions of the TSK fuzzy PI control system. And we use LMI to check if these conditions can be found. In the future work, this stability analysis method will be developed for more general TSK fuzzy control systems such as TSK fuzzy PID control system and the given plant will be extended to a certain class of nonlinear system.

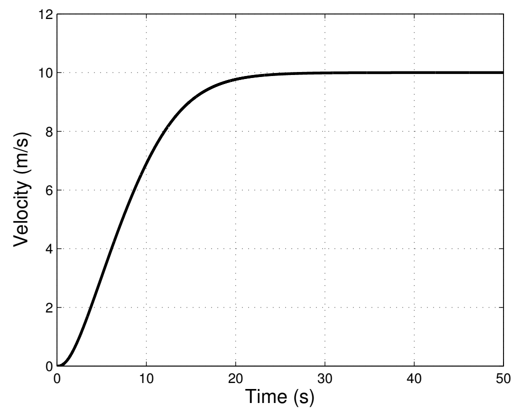


Fig. 7: Vehicle speed of the automobile cruise control system.

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