Single-input Single-output Convex Fuzzy Systems as Universal Approximators for Single-input Single-output Convex Functions

Jinwook Kim and Jin S. Lee

Abstract— **This paper derives sufficient conditions for a single-input single-output fuzzy system whose output is convex with respect to its input. They are composed of two parts: the constraint on input membership functions and the constraint on the THEN parts of fuzzy rule base. And we prove in a constructive manner that any continuously differentiable single-input single-output convex function can be approximated with any degree of accuracy by a single-input single-output fuzzy system with the above conditions.**

I. Introduction

We often encounter the real problems that deal with the convex input-output relationship. In the field of *Computer Aided Geometric Design*, convexity preserving interpolation [1]-[4] is popular and practically important in modeling the surface of the objects. In the car body modeling as an example, its surface is required to be convex (or concave) and contain no wiggles. Another example is a diode, one of the elementary nonlinear electronic elements [5]. In the forward-bias operation region, the i-v relationship of a diode is approximated by $i = I_s(e^{v/nV_T} - 1)$ where I_s is the saturation current, V_T is the thermal voltage and n is the constant that depends on the material and the diode structure. Because the i-v relationship is roughly represented by an exponential form, it shows that a diode has the convexity between the given current i and the voltage v. In [6], S. Osowski used a piecewise-linear convex model for the diode, which is important from the practical point of view. Likewise, when we model a diode using the fuzzy model, it is desired that the fuzzy model preserves the convex relationship between the voltage (input) and the current (output). This type of fuzzy system whose output is convex with respect to its input is called a convex fuzzy system (CFS) [7].

In recent years, researchers stated to consider the convexity in a fuzzy system. In [8], Park developed sufficient conditions that realize a single-input single-output (SISO) CFS. Using a modified gradient descent-based adaptation algorithm, he formulated a fuzzy parameter identification method so that the CFS converges to the given convex target function. In [7], Kim et al. derived sufficient conditions for the zeroth-order and for the first-order SISO convex Takagi-Sugeno-Kang (TSK) fuzzy systems and formulated a least squares (LS) based identification method for a convex function. In these papers, the CFSs are shown to perform better than conventional fuzzy systems in robustness properties to insufficient and noisy data pairs. In this paper, we derive sufficient conditions on input fuzzy membership functions and the THEN parts of fuzzy rule base for SISO CFS.

From the above results, an important question arises: Can a given type of SISO CFS approximate a SISO convex function? If a given CFS is identified as the universal approximator to a convex function, then it will make a good model in the practical systems that have convex input-output relationship. Many papers and books prove that the fuzzy system is a universal approximator to any continuous function or smooth function (e.g., [9]-[22]). For example, Wang [9] proved that a fuzzy system with gaussian membership functions, center average defuzzifier and product inference engine is able to serve as an universal approximator to any continuous function. Using polynomial and Weierstrass approximation theorem [23], H. Ying [19] proved that a Takagi-Sugeno-Kang (TSK) type fuzzy system is also a universal approximator to any continuous function. In this paper, we introduce a constructive manner how to design a fuzzy system constrained by the convexity and prove that the generated SISO CFS can approximate the continuously differentiable SISO convex function.

This paper is organized as follows: Section II suggests sufficient conditions for a fuzzy system to have the convex input-output relationship. Section III show that SISO CFS can serve as universal approximator of any continuously differentiable SISO convex function. Section IV gives the examples to design SISO CFS and Section V makes conclusions.

II. Single-input Single-output Convex Fuzzy Systems

Some definitions are given to formally describe a CFS.

Definition 1: (*Convex Set* [24]) A set U is *convex* if the line segment between any two points in U lies in U, *i.e.*, if for any $x_1, x_2 \in U$ and any λ with $0 \le \lambda \le 1$, we have

$$\lambda x_1 + (1-\lambda)x_2 \in U.$$

Definition 2: (*Convex Function* [24]) A function f is *convex* if U is convex set and if for any $x_1, x_2 \in U$, and λ

Jinwook Kim is with the Department of Electrical Engineering, Pohang University of Science and Technology, Pohang, Korea (phone: +82 54 279 5574; email: tw0822@postech.ac.kr).

Jin S. Lee is with the Department of Electrical Engineering, Pohang University of Science and Technology, Pohang, Korea (phone: +82 54 279 5574; email: jsoo@postech.ac.kr).

with $0 \leq \lambda \leq 1$, we have

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2).$$

If a fuzzy system satisfies the definition of convex function, it is called a CFS.

Consider a SISO fuzzy system whose IF-THEN rules are of the following form:

$$R^l: \text{ IF } x \text{ is } A^l \text{ THEN } y \text{ is } B^l \tag{1}$$

where A^l and B^l are finite complete [9] fuzzy sets in U and $1 \leq l \leq M$ is an index for rule base.

The output of a SISO fuzzy system with singleton fuzzifier and center average defuzzifier is given as

$$y = f(x) = \frac{\sum_{l=1}^{M} \bar{y}^l \cdot \mu_{A^l}(x)}{\sum_{l=1}^{M} \mu_{A^l}(x)} \tag{2}$$

where $\mu_{A^l}(x)$ is the membership function that characterizes the fuzzy set A^l and $\bar{y}^l$, what we call a consequent part parameter, is the center of B^l.

We now derive sufficient conditions for the SISO fuzzy system to become convex [7], [8]. Here, the notations and shape of the triangular membership function used in this paper are depicted in Fig. 1.

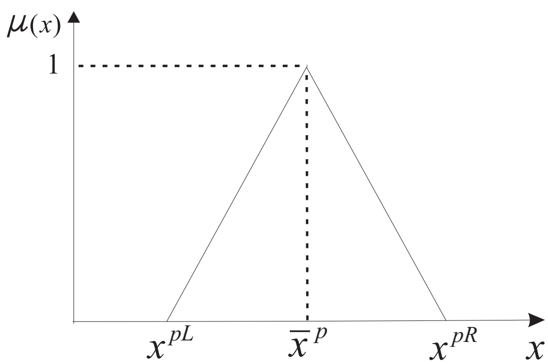

Fig. 1. The pth triangular membership function and its notations.

Theorem 1: (*Sufficient conditions for a SISO CFS*) The output of the fuzzy system (2) with triangular membership functions is convex with respect to x if the following conditions are satisfied:

1) $\bar{x}^p < \bar{x}^q$ for $1 \leq p < q \leq M$
2) $\bar{x}^p = x^{(p+1)L}$ and $\bar{x}^{p+1} = x^{pR}$ for $1 \leq p \leq M-1$
3) $\bar{y}^p \leq \frac{\bar{x}^{p+1}-\bar{x}^p}{\bar{x}^{p+1}-\bar{x}^{p-1}} \bar{y}^{p-1} + \frac{\bar{x}^{p}-\bar{x}^{p-1}}{\bar{x}^{p+1}-\bar{x}^{p-1}} \bar{y}^{p+1}$ for $2 \leq p \leq M-1$

Proof: If $\bar{x}^p < x < \bar{x}^{p+1}$ for $1 \leq p \leq M-1$, then input x is covered by only two triangular membership functions $\mu_{A^p}(x)$ and $\mu_{A^{p+1}}(x)$. That is, $\mu_{A^r}(x) = 0$ for $r \neq p$ and $r \neq p+1$ (Fig. 2).
Thus (2) is changed to

$$\begin{aligned} f(x) &= \frac{\sum_{l=p}^{p+1} \bar{y}^l \cdot \mu_{A^l}(x)}{\sum_{l=p}^{p+1} \mu_{A^l}(x)} \\ &= \frac{\bar{y}^p \cdot \mu_{A^p}(x) + \bar{y}^{p+1} \cdot \mu_{A^{p+1}}(x)}{\mu_{A^p}(x) + \mu_{A^{p+1}}(x)}. \end{aligned} \tag{3}$$

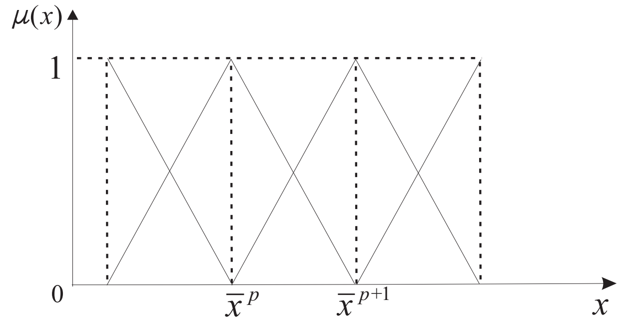

Fig. 2. Membership functions on pth and $(p+1)$th rules.

When $\bar{x}^p < x < \bar{x}^{p+1}$, $\mu^p(x)$ and $\mu^{p+1}(x)$ are given respectively by

$$\mu_{A^p}(x) = \frac{\bar{x}^{p+1} - x}{\bar{x}^{p+1} - \bar{x}^p} \tag{4}$$

and

$$\mu_{A^{p+1}}(x) = \frac{x - \bar{x}^p}{\bar{x}^{p+1} - \bar{x}^p}. \tag{5}$$

Inserting (4) and (5) to (3) gives

$$\begin{aligned} f(x) &= \frac{\bar{y}^p \cdot \frac{\bar{x}^{p+1}-x}{\bar{x}^{p+1}-\bar{x}^p} + \bar{y}^{p+1} \cdot \frac{x-\bar{x}^p}{\bar{x}^{p+1}-\bar{x}^p}}{\frac{\bar{x}^{p+1}-x}{\bar{x}^{p+1}-\bar{x}^p} + \frac{x-\bar{x}^p}{\bar{x}^{p+1}-\bar{x}^p}} \\ &= \frac{\bar{y}^p \cdot \left(\bar{x}^{p+1} - x\right) + \bar{y}^{p+1} \cdot (x - \bar{x}^p)}{\bar{x}^{p+1} - \bar{x}^p} \\ &= \frac{\left(\bar{y}^{p+1} - \bar{y}^p\right) x + \bar{y}^p \bar{x}^{p+1} - \bar{y}^{p+1} \bar{x}^p}{\bar{x}^{p+1} - \bar{x}^p} \\ &= \frac{\bar{y}^{p+1} - \bar{y}^p}{\bar{x}^{p+1} - \bar{x}^p} x + \frac{\bar{y}^p \bar{x}^{p+1} - \bar{y}^{p+1} \bar{x}^p}{\bar{x}^{p+1} - \bar{x}^p} \\ &= ax + b, \end{aligned} \tag{6}$$

where

$$a = \frac{\bar{y}^{p+1} - \bar{y}^p}{\bar{x}^{p+1} - \bar{x}^p}$$

and

$$b = \frac{\bar{y}^p \bar{x}^{p+1} - \bar{y}^{p+1} \bar{x}^p}{\bar{x}^{p+1} - \bar{x}^p}.$$

Once $\bar{x}^p, \bar{x}^{p+1}, \bar{y}^p$ and $\bar{y}^{p+1}$ are given, then a and b become constants. Thus, for $1 \leq p \leq M-1$, $f(x)$ is a straight line function with respect to x between $\bar{x}^p$ and $\bar{x}^{p+1}$ (Fig. 3).

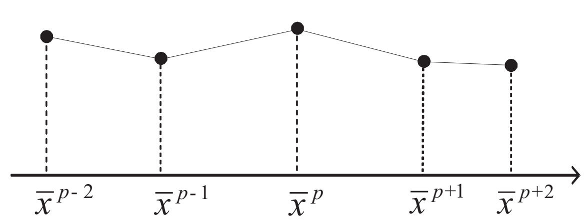

Fig. 3. Straight line function between adjacent points.

If $x = \bar{x}^p$, then $\mu_{A^r}(x) = 1$ for $r = p$ and $\mu_{A^r}(x) = 0$ for $r \neq p$. Thus (2) is given by

$$f(\bar{x}^p) = \bar{y}^p.$$

To be convex, the set of points $(\bar{x}^p, \bar{y}^p)$ for $1 \leq p \leq M$ must satisfy the following property (Fig. 4) [25]:

$$slope\left(\overline{AB}\right) \leq slope\left(\overline{AC}\right) \leq slope\left(\overline{BC}\right).$$

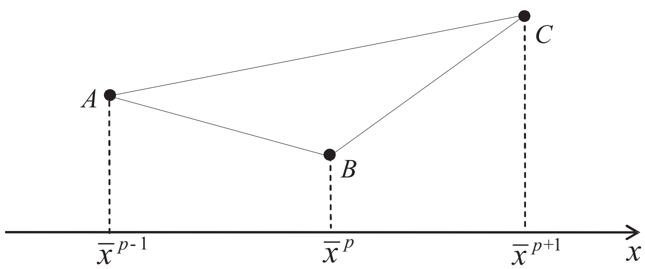

Fig. 4. Three vertices in three-cord property.

By using the relationship between $slope\left(\overline{AB}\right)$ and $slope\left(\overline{BC}\right)$, the following inequality can be derived:

$$\frac{f(\bar{x}^p) - f(\bar{x}^{p-1})}{\bar{x}^p - \bar{x}^{p-1}} \leq \frac{f(\bar{x}^{p+1}) - f(\bar{x}^p)}{\bar{x}^{p+1} - \bar{x}^p}$$

$$\frac{\bar{y}^p - \bar{y}^{p-1}}{\bar{x}^p - \bar{x}^{p-1}} \leq \frac{\bar{y}^{p+1} - \bar{y}^p}{\bar{x}^{p+1} - \bar{x}^p}$$

$$\bar{y}^p \leq \frac{\bar{x}^{p+1} - \bar{x}^p}{\bar{x}^{p+1} - \bar{x}^{p-1}}\bar{y}^{p-1} + \frac{\bar{x}^p - \bar{x}^{p-1}}{\bar{x}^{p+1} - \bar{x}^{p-1}}\bar{y}^{p+1}. \quad (7)$$

From (6) and (7), we can see that $f(x)$ in $\bar{x}^{p-1} \leq x \leq \bar{x}^{p+1}$ is a part of convex polygon whose vertices are $\bar{y}^{p-1}, \bar{y}^p$ and $\bar{y}^{p+1}$. Because every three adjacent vertices $\bar{y}^{p-1}, \bar{y}^p$ and $\bar{y}^{p+1}$ for $2 \leq p \leq M-1$ constitute the convex polygon, the whole polygon is also convex and piecewise linear (or affine) as shown in Fig. 5.

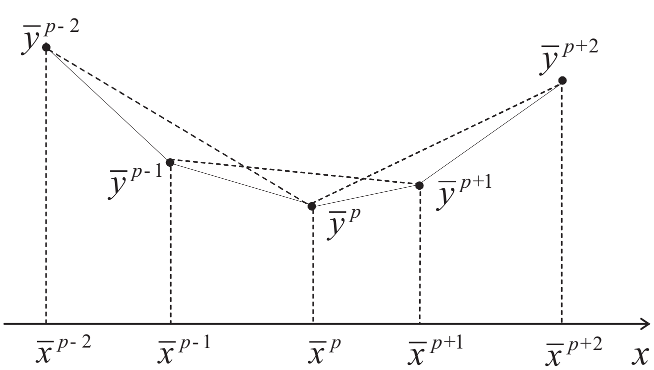

Fig. 5. $f(x)$ as a part of a convex polygon.

■

III. Universal Approximator of SISO Convex Function

Several results (e.g., [9]-[22]) show that a fuzzy system can approximate any continuous or smooth function on a compact set to an arbitrary degree of accuracy. In this section, we prove in a constructive manner that the developed SISO CFS can approximate any continuously differentiable SISO convex function on a compact set to any required accuracy. In developing a SISO CFS, we first need information available on the target function $g(x) : U \subset \mathbb{R} \to V \subset \mathbb{R}$. In practical problems, the mathematical model of target function $g(x)$ is usually unknown but we have input-output data pairs $(x^j, g(x^j))$ where $j = 1, \cdots, r$ and r is the number of data pairs [9]. In this section, it is assumed that the output of the target function is convex with respect to its input. Based on this priori information about $g(x)$, we construct a SISO CFS as follows.

1) Define M fuzzy sets $A^1, \cdots, A^M$ in $[\alpha, \beta] \subset U$, which are characterized by triangular membership functions with conditions 1) and 2) in Theorem 1. And define $\alpha = \bar{x}^1$ and $\beta = \bar{x}^M$. The triangular membership functions are as follows:

$$\mu_{A^1}(x) = \mu_{A^1}(x; \bar{x}^1, \bar{x}^1, \bar{x}^2),$$
$$\mu_{A^p}(x) = \mu_{A^p}(x; \bar{x}^{p-1}, \bar{x}^p, \bar{x}^{p+1})$$

for $2 \leq p \leq M-1$ and

$$\mu_{A^M}(x) = \mu_{A^M}(x; \bar{x}^{M-1}, \bar{x}^M, \bar{x}^M).$$

2) Construct IF-THEN rules for the SISO CFS as follows:

R^l : IF x is A^l THEN y is B^l

where $1 \leq l \leq M$ (same as (1)). Set the consequent part parameters to

$$\bar{y}^p = f\left(\bar{x}^p\right) = g\left(\bar{x}^p\right). \quad (8)$$

Because $g(x)$ is convex, it satisfies condition 3) in Theorem 1. In fact, for $\left(\bar{x}^{p-1}, \bar{y}^{p-1}\right)$, $\left(\bar{x}^p, \bar{y}^p\right)$ and $\left(\bar{x}^{p+1}, \bar{y}^{p+1}\right)$, we have

$$\begin{aligned} g\left(\bar{x}^p\right) &= \bar{y}^p \\ &\leq \frac{\bar{x}^{p+1} - \bar{x}^p}{\bar{x}^{p+1} - \bar{x}^{p-1}}\bar{y}^{p-1} + \frac{\bar{x}^p - \bar{x}^{p-1}}{\bar{x}^{p+1} - \bar{x}^{p-1}}\bar{y}^{p+1} \\ &= \frac{\bar{x}^{p+1} - \bar{x}^p}{\bar{x}^{p+1} - \bar{x}^{p-1}} g\left(\bar{x}^{p-1}\right) + \frac{\bar{x}^p - \bar{x}^{p-1}}{\bar{x}^{p+1} - \bar{x}^{p-1}} g\left(\bar{x}^{p+1}\right). \end{aligned}$$

for $2 \leq p \leq M-1$.

3) Through the previous steps, we can construct a fuzzy system with the form of (2). Because it satisfies all conditions in Theorem 1, it is the SISO CFS.

Now we prove that the constructed SISO CFS can serve as a universal approximator to any continuously differentiable SISO convex function.

Theorem 2: Let $f(x)$ be the constructed SISO CFS and $g(x)$ be the target function with the convexity. If $g(x)$ is continuously differentiable on $U = [\alpha, \beta] \subset \mathbb{R}$, then

$$\|g - f\|_\infty \leq \left\|\frac{\partial g}{\partial x}\right\|_\infty h$$

where $\|d(x)\|_\infty = \sup_{x\in U} |d(x)|$ for any function $d(x)$ and $h = \max_{1\le p\le M-1} \left|\bar{x}^{p+1} - \bar{x}^p\right|$.

Proof: Because fuzzy sets used in $f(x)$ are characterized by triangular membership functions satisfying conditions 1) and 2) in Theorem 1, input x is covered by at most two triangular membership functions $\mu_{A^p}(x)$ and $\mu_{A^{p+1}}(x)$ as shown in Fig. 2. Thus (2) is changed to

$$f(x) = \frac{\sum_{l=p}^{p+1} \bar{y}^l \cdot \mu_{A^l}(x)}{\sum_{l=p}^{p+1} \mu_{A^l}(x)} = \sum_{l=p}^{p+1} \left[\frac{\mu_{A^l}(x)}{\sum_{l=p}^{p+1} \mu_{A^l}(x)}\right] \bar{y}^l$$

and by using (8), we have

$$f(x) = \sum_{l=p}^{p+1} \left[\frac{\mu_{A^l}(x)}{\sum_{l=p}^{p+1} \mu_{A^l}(x)}\right] \bar{g}\left(\bar{x}^l\right).$$

Because

$$\sum_{l=p}^{p+1} \left[\frac{\mu_{A^l}(x)}{\sum_{l=p}^{p+1} \mu_{A^l}(x)}\right] = 1,$$

we have

$$\begin{aligned} &|g(x) - f(x)| \\ &\le \sum_{l=p}^{p+1} \left[\frac{\mu_{A^l}(x)}{\sum_{l=p}^{p+1} \mu_{A^l}(x)}\right] \left|g\left(x\right) - g\left(\bar{x}^l\right)\right| \\ &\le \max_{l=p,p+1} \left|g\left(x\right) - g(\bar{x}^l)\right|. \end{aligned}$$

From the Mean Value Theorem, we derive the following inequality:

$$|g(x) - f(x)| \le \max_{l=p,p+1} \left\|\frac{\partial g}{\partial x}\right\|_\infty \left|x - \bar{x}^l\right|. \tag{9}$$

Because $x \in \left[\bar{x}^p, \bar{x}^{p+1}\right]$, we know that $|x - \bar{x}^p| \le \left|\bar{x}^{p+1} - \bar{x}^p\right|$. Hence, (9) becomes

$$|g(x) - f(x)| \le \max_{1\le p\le M-1} \left\|\frac{\partial g}{\partial x}\right\|_\infty \left|\bar{x}^{p+1} - \bar{x}^p\right|.$$

From the above inequality, we have

$$\begin{aligned} \|g(x) - f(x)\|_\infty &\le \left\|\frac{\partial g}{\partial x}\right\|_\infty \max_{1\le p\le M-1} \left|\bar{x}^{p+1} - \bar{x}^p\right| \\ &\le \left\|\frac{\partial g}{\partial x}\right\|_\infty h \end{aligned} \tag{10}$$

where $h = \max_{1\le p\le M-1} \left|\bar{x}^{p+1} - \bar{x}^p\right|$. In (10), $\left\|\frac{\partial g}{\partial x}\right\|_\infty$ is a finite number and we can adjust the value of h by increasing or decreasing the number of input fuzzy sets. That is, $\|g(x) - f(x)\|_\infty \le \epsilon$. ■

Theorem 2 shows that the constructed SISO CFS can approximate any continuously differentiable SISO convex function on a compact set U to a required error bound ϵ. But, in computing the required accuracy, we need to know the bound of the derivative of the target function with respect to its input.

IV. Simulation Examples

A. Convex Function Approximation

We give an example that shows how to generate a SISO CFS based on Theorem 1 and Theorem 2. As a target function, we select the following SISO convex function:

$$g(x) = x^2 \tag{11}$$

where $x \in U = [-1, 1]$. Because $\frac{\partial^2 g}{\partial x^2} = 2 > 0$ in $\mathbb{R}$, it is convex in U. To use the result of Theorem 2, we assume that we can know the bound of derivative of given target function with respect to its input.

$$\left\|\frac{\partial g}{\partial x}\right\|_\infty = 2$$

in U.

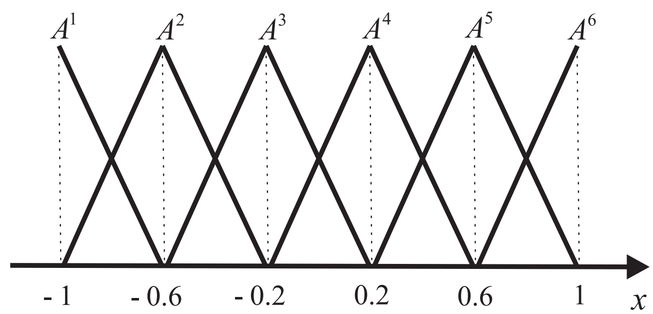

Fig. 6. Triangular membership functions to implement SISO CFS when $\epsilon = 0.8$.

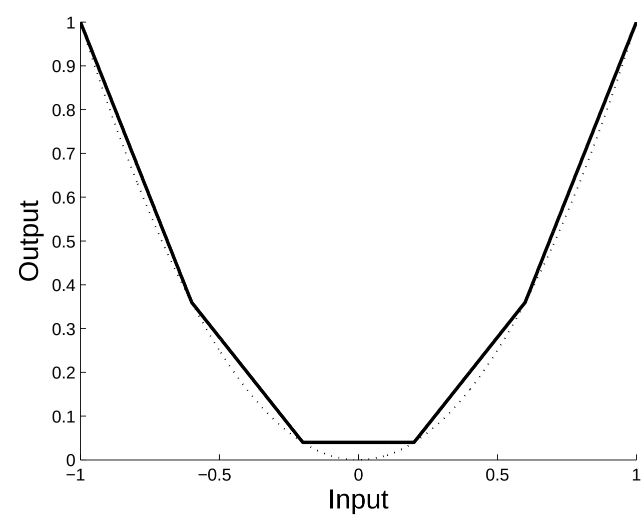

Fig. 7. The graphs of target function (dotted line) and SISO CFS (solid line) when $\epsilon = 0.8$.

If $\epsilon = 0.8$, $\|g(x) - f(x)\|_\infty \le \left\|\frac{\partial g}{\partial x}\right\|_\infty h = 2h = 0.8$. So $h = 0.4$ meets the requirement and we need 6 fuzzy sets. In developing a SISO CFS, we use triangular membership functions as shown in Fig. 6 and arrange their center $\bar{x}^p$ at $-1.4 + p \times 0.4$ where $1 \le p \le 6$. And we set the consequent

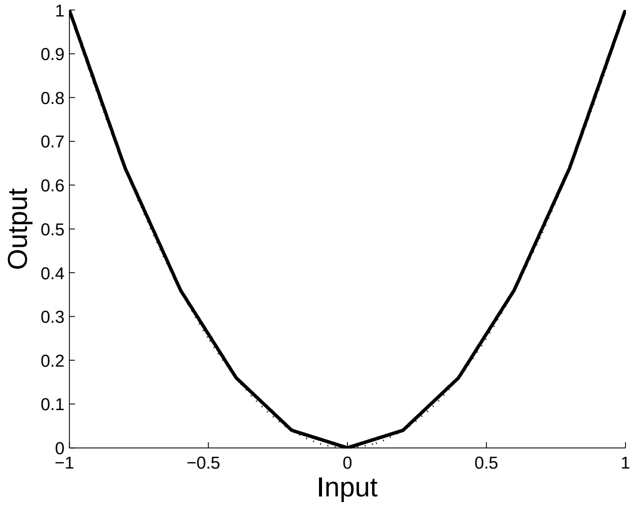

Fig. 8. The graphs of target function (dotted line) and SISO CFS (solid line) when $\epsilon = 0.4$.

parameter $\bar{y}^p$ to $g(\bar{x}^p)$ to satisfy condition 3) in Theorem 1. Then the developed SISO CFS is given by

$$y = f(x) = \frac{\sum_{l=1}^{6} g\left(\bar{x}^l\right) \cdot \mu_{A^l}(x)}{\sum_{l=1}^{6} \mu_{A^l}(x)}$$

whose graph is given in Fig. 7. The result graph is convex with respect to x. However, it shows the large gap between target function and SISO CFS due to the linearity of triangular membership function. To approximate $g(x)$ with a smooth CFS, we may reduce ϵ by using more fuzzy sets in x axis. For example, if $\epsilon = 0.4$, then $h = 0.2$ meets the requirement and we need 11 fuzzy sets for input space. Through the same procedure in the case of $\epsilon = 0.8$, we arrange the triangular membership functions and set the consequent parameters. Then, the developed CFS is shown in Fig. 8, which looks almost the same as the target function.

B. Comparison between CFS and Conventional Fuzzy System

In this section, we compare CFSs with conventional fuzzy systems thorough simple identification example. We select (11) as a target function. We arrange 6 membership functions for input space as shown in Fig. 6 and select data pairs from the target function. To consider noise effect as in practical situations, we add random noise uniformly distributed in $[-0.3, 0.3]$ to output data. Then the output of the developed fuzzy system is given by

$$y = f(x) = \frac{\sum_{l=1}^{6} \bar{y}^l \cdot \mu_{A^l}(x)}{\sum_{l=1}^{6} \mu_{A^l}(x)}.$$

To determine $\bar{y}^l$, we use LS [7]. We compare two methods. One is the conventional least squares (LS) and the other is the constrained LS by condition 3) in Theorem 1. The fuzzy system trained by the latter method is SISO CFS. When the

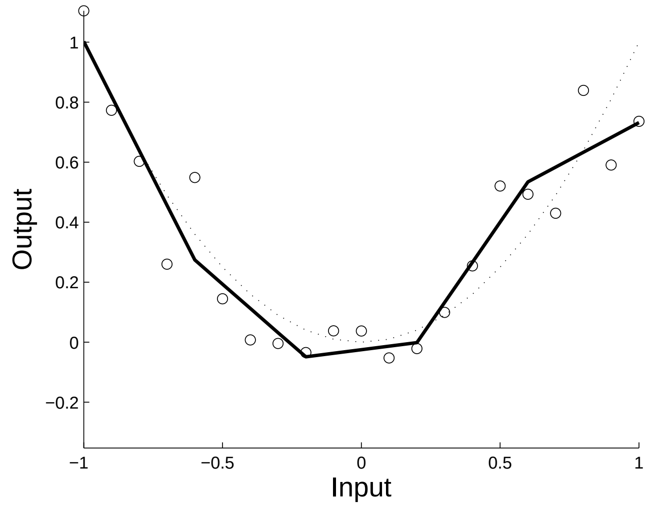

Fig. 9. The identification result of conventional fuzzy system (black line: identification result, dotted line: target function, circle: noisy input-output pairs).

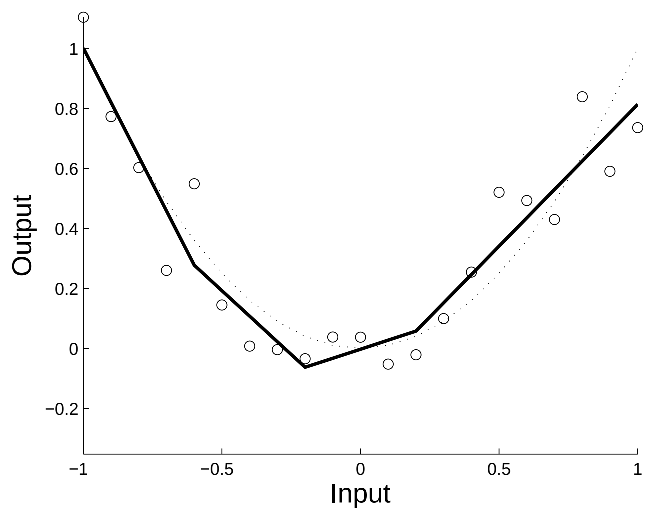

Fig. 10. The identification result of CFS (black line: identification result, dotted line: target function, circle: noisy input-output pairs).

number of data pairs is 21, the results are shown in Fig. 9 and Fig. 10.

The root mean square error (RMSE) between the identification results and the target function are given in TABLE I. From the results, the developed CFSs are shown to perform

TABLE I

AVERAGE RMSE OBTAINED FROM 10 EVALUATIONS USING THE DIFFERENT NUMBER OF DATA SET WITH TWO FUZZY SYSTEMS

System	The number of data		
	6	11	21
Convex Fuzzy System	0.3233	0.2936	0.3668
Conventional Fuzzy System	0.3424	0.3412	0.4035

better than the conventional fuzzy systems. It is mainly because the conventional fuzzy systems are over-fitted to the given noisy data pairs, but the CFSs are not just fitted to the data due to the convexity.

V. Conclusions

In this paper, we derived sufficient conditions for a fuzzy system to guarantee the convex input-output relationship. Suggested conditions are derived for a SISO case. And the conditions are composed of two parts: the constraint on input membership functions and the constraint on the consequent parameters. A fuzzy system constrained by these conditions is of a piecewise linear (or affine) convex model. And we showed the approximation property of SISO CFS for continuously differentiable SISO convex function. Through simulation examples, we developed the SISO CFS for approximating the target function with convexity and showed that the developed CFS are better than a conventional fuzzy system in the convex function identification.

ACKNOWLEDGMENT

This research was financially supported by the Ministry of Education, Science Technology (MEST) and Korea Industrial Technology Foundation (KOTEF) through the Human Resource Training Project for Regional Innovation.

References

[1] Kuijt, F., *Convexity preserving interpolation; stationary nonlinear subdivision and splines*, Ph.D. Thesis, Eindhoven University of Technology, Eindhoven, The Netherlands, 1998.

[2] X.Q. Yang and C.J. Goh, “A method for convex curve approximation,” *European Journal of Operational Research*, vol. 97, no. 1, pp. 205-212, 1997.

[3] Aidi Li, “Convexity preserving interpolation,” *Computer Aided Geometric Design*, vol. 16, no. 2, pp. 127-147, 1999.

[4] Dick den Hertog, Etienne de Klerk and Kees Roos, “On convex quadratic approximation,” *Statistica Neerlandica*, vol. 56, no. 3, pp. 376-385, 2002.

[5] Adel S. Sedra and Kenneth C. Smith, *Microelectronic Circuits*, Oxford University Press, 1998.

[6] S. Osowski, “Optimisation approach to the analysis of piecewise-linear convex circuits,” *IEE Proceding-G*, vol. 130, no. 3, pp. 295-300, Jun. 1992.

[7] Tae-Wook Kim, Sang Y. Park and Jin S. Lee, “Parameter conditions and least squares identification of single-input single-output convex fuzzy system,” in *SICE Annual Conference 2007*, Kagawa, Japan, Sept. 2007, pp. 568-573.

[8] Sang Y. Park, *Constrained Optimization of Fuzzy Logic Systems with its Application to Current Prediction for Automatic Crane Operations*, Ph.D Thesis, Pohang University of Science and Technology, 1999.

[9] L.X. Wang, *A Course in Fuzzy Systems and Control*, Englewood Cliffs, Prentice-Hall, 1997.

[10] L.X. Wang, and Mendel, J.M., “Fuzzy basis functions, universal approximation, and orthogonal least square learning,” *IEEE trans. Neural Netw.*, vol. 3, no. 5, pp. 807-814, Sep. 1992.

[11] J. J. Buckley, “Sugeno type controllers are universal controllers,” *Fuzzy Sets Syst.*, vol. 53, no. 3, pp. 299-303, Feb. 1993.

[12] J. L. Castro and M. Delgado, “Fuzzy systems with defuzzification are universal approximators,” *IEEE Trans. Syst. Man Cybern. B, Cybern.*, vol. 26, no. 1, pp. 149-152, Feb. 1996.

[13] B. Kosko, “Fuzzy systems as universal approximators,” in *Proc. IEEE Int. Conf. Fuzzy Syst.*, San Diego, CA, Mar. 1992, pp. 8-12.

[14] F. L. Lewis, S.-Q. Zhu, and K. Liu, “Function approximation by fuzzy systems,” in *Proc. American Control Conf.*, Seattle, WA, Jun. 1995, pp. 3760-3764.

[15] H. T. Nguyen, V. Kreinovich, and O. Sirisaengtaksin, “Fuzzy control as a universal control tool,” *Fuzzy Sets Syst.*, vol. 80, no. 1, pp. 71-86, May 1996.

[16] L.-X. Wang, “Fuzzy systems are universal approximators,” in *Proc. IEEE Int. Conf. Fuzzy Syst.*, San Diego, CA, Mar. 1992, pp. 1163-1170.

[17] H. Ying, “Sufficient conditions on general fuzzy systems as function approximators,” *Automatica*, vol. 30, no. 3, pp. 521-525, Mar. 1994.

[18] H. Ying and G.-R. Chen, “Necessary conditions for some typical fuzzy systems as universal approximators,” *Automatica*, vol. 33, no. 7, pp. 1333-1338, Jul. 1997.

[19] H. Ying, “Sufficient conditions on uniform approximation of multivariate functions by general takagi-sugeno fuzzy systems with linear rule consequent,” *IEEE Trans. Syst. Man Cybern. A, Syst. Humans.*, vol. 28, no. 4, pp. 515-520, Jul. 1998.

[20] X.-J. Zeng and M. G. Singh, “Approximation theory of fuzzy systems-SISO case,” *IEEE Trans. Fuzzy Syst.*, vol. 2, no. 2, pp. 162-176, May 1994.

[21] X.-J. Zeng and M. G. Singh, “Approximation theory of fuzzy systems-MIMO case,” *IEEE Trans. Fuzzy Syst.*, vol. 3, no. 2, pp. 219-235, May 1995.

[22] Vladik Kreinovich, Hung T. Nguyen, and Yeung Yam, “Fuzzy Systems Are Universal Approximators for a Smooth Function And Its Derivative,” *Int. J. Intell. Syst.*, vol. 15, no. 6, pp. 565-574, 2000.

[23] Rudin, *Principles of mathematical analysis*, McGraw-Hill, 1976.

[24] Stephen Boyd and Lieven Vandenberghe, *Convex Optimization*, Chpater 2 and 3, Cambridge University Press, 2004.

[25] J.-B. Hiriart-Urruty and C. Lemarechal, *Convex Analysis and Minimization Algorithms I*, volume 305 of Grundlehren der mathematischen Wissenschaften. Springer-Verlag, New York, 1993.