Design of Fuzzy System with Linear Bounds

Sengjae Cho and Jin S. Lee
Department of Electronic and Electrical Engineering
Pohang University of Science and Technology, Korea
Email: {sjcho80, jsoo}@postech.ac.kr

***Abstract*—In this paper, we derive sufficient conditions under which the single-input single-output fuzzy system is bounded by linear functions in its output. If the fuzzy system is constructed with the complete and consistent fuzzy sets with trapezoid membership functions, we show that the derived conditions are simply the constraints on the consequent part parameters. Then we prove that any continuously differentiable and linearly bounded function in its output can be approximated with any degree of accuracy by the fuzzy system formed with the above fuzzy sets and conditions.**

I. Introduction

For many real world engineering problems, we often encounter a system bounded by linear functions in its output. For example, bounds on the output of an amplifier can be modeled by a saturation function that consists of linear functions [1]. This type of fuzzy system whose output is bounded by linear functions is called the linearly bounded fuzzy system (LBFS).

An important question to investigate is whether a given type of LBFS is able to approximate any continuously differentiable and linearly bounded function. If a given LBFS is identified as a universal approximator to these functions, then it will make a good model for the practical systems that comes with linear bounds. Many papers and books report that the fuzzy systems are universal approximators to any continuously differentiable function (e.g. [2]-[8]). For example, Wang [2] proved that a fuzzy system with trapezoid membership functions, center average defuzzifier and product inference engine is able to serve as a universal approximator to any continuously differentiable function. In many results of these papers, the approximation property has been proved using the Stone-Weierstrass [9] theorem and, as a result, they just established the existence results of a fuzzy system that approximates a given continuously differentiable function with any degree of accuracy.

In this paper, we introduce a constructive method to generate an LBFS and prove that the generated LBFS can serve as a universal approximator to a continuously differentiable and linearly bounded function.

This paper is organized as follows: In Section II, we derive sufficient conditions to construct an LBFS. In Section III, we show that the LBFS developed in Section II serve as a universal approximator of any continuously differentiable and linearly bounded function. Section IV gives simulation examples and Section V makes conclusions.

II. Linearly Bounded Fuzzy System

For single-input single-output fuzzy system, the involved IF-THEN rules of a fuzzy system are of the following form:

$$R^l: \text{ IF } x \text{ is } A^l, \text{ THEN } y \text{ is } B^l \tag{1}$$

where A^l and B^l are fuzzy sets and $1 \leq l \leq M$ is an index for the input fuzzy set associated with input $x \in [\alpha, \beta]$. If we use singleton fuzzifier and center average defuzzifier, the output of the fuzzy system we use in the sequel can be formulated as

$$f(x) = \frac{\sum_{l=1}^{M} \overline{y}^l \mu^l(x)}{\sum_{l=1}^{M} \mu^l(x)}, \tag{2}$$

where $\mu^l(x)$ is the membership function of A^l and $\overline{y}^l$ is the center of B^l.

To characterize the fuzzy sets and membership functions we use for (1) and (2), we introduce the following definitions [2].

Definition 1 (Trapezoid membership function):

$$\mu^l(x : a^l, b^l, c^l, d^l) = \begin{cases} 0 & \text{if } x \leq a^l \\ \frac{x-a^l}{b^l-a^l} & \text{if } a^l \leq x \leq b^l \\ 1 & \text{if } b^l \leq x \leq c^l \\ \frac{d^l-x}{d^l-c^l} & \text{if } c^l \leq x \leq d^l \\ 0 & \text{if } d^l \leq x \end{cases}$$

In addition, we define $\beta = a^{M+1}$ and $\alpha = d^0$.

Definition 2 (Completeness): Fuzzy sets $A^1, ..., A^M$ in $U \subset \mathbb{R}$ are said to be *complete* on U if for any $x \in U$, there exists a fuzzy set A^l such that $\mu^l(x) > 0$.

Definition 3 (Consistency): Fuzzy sets $A^1, ..., A^M$ in $U \subset \mathbb{R}$ are said to be *consistent* on U if $\mu^l(x) = 1$ for some $x \in U$, then $\mu^k(x) = 0$ for all $l \neq k$.

An LBFS is a fuzzy system defined as following:

Definition 4 (Linearly Constrained Fuzzy System): Let $x \in U = [\alpha, \beta] \subset \mathbb{R}$ be an input to the function $f(x) \in V \subset \mathbb{R}$ and

$$h(x) = \pi_1 x + \pi_0 \tag{3}$$

be a bound of $f(x)$. Then, $f : U \to V$ is said to be linearly bounded if $f(x) \leq h(x)$ (or $f(x) \geq h(x)$) for all $x \in U$.

We can now derive sufficient conditions on the consequent parameters of the fuzzy system to become an LBFS.

Theorem 1: Let the fuzzy system $f(x)$ as given in (2) be represented with complete and consistent fuzzy sets that are characterized by the trapezoid membership functions. Then $f(x) \geq h(x)$ if

$$\overline{y}^l \geq h(\overline{x}^l), \text{ where } \overline{x}^l =: \begin{cases} a^{l+1} & \text{if } \pi_1 > 0 \\ d^{l-1} & \text{if } \pi_1 \leq 0 \end{cases}$$

for all $1 \leq l \leq M$.

Proof: See Appendix A. ■

Remark 1: When $f(x) \leq h(x)$, the sufficient condition in Theorem 1 is equivalent to

$$-f(x) = \frac{\sum_{l=1}^{M} (-\overline{y}^l)\mu^l(x)}{\sum_{l=1}^{M} \mu^l(x)} \geq -h(x) = (-\pi_1)x + (-\pi_0)$$

Hence, $f(x) \leq h(x)$ if

$$\overline{y}^l \leq h(\overline{x}^l), \text{ where } \overline{x}^l =: \begin{cases} a^{l+1} & \text{if } \pi_1 < 0 \\ d^{l-1} & \text{if } \pi_1 \geq 0 \end{cases}$$

for all $1 \leq l \leq M$.

III. Universal approximation properties

Several papers and books [2]-[8] show that a fuzzy system can approximate any continuously differentiable functions on a compact set to an arbitrary degree of accuracy. However, these papers do not consider the case that the fuzzy output is linearly bounded. For an example as given in Sec. 10. 2 in [2], design of a fuzzy system f that shows universal approximation property does not guarantee $f \geq h$ if $x \in [(b^l + c^l)/2, a^{l+1}]$ and $\pi_1 \geq 0$. In this section, we prove in a constructive manner that the developed LBFS can approximate any continuously differentiable and linearly bounded function on a compact set to any required accuracy. In developing an LBFS, we first need information available on the target function $g(x) : U \to V \subset \mathbb{R}$. In practical problems, the target function $g(x)$ is usually unknown but we have input-output data pairs $(x, g(x))$ available [2]. In addition, it is also known that the output of the target function is linearly bounded: that is, $g(x) \geq h(x)$ (or $g(x) \leq h(x)$). Based on this information about $g(x)$, we construct an LBFS as follows:

1) Define M fuzzy sets $A^1, ..., A^M$ in $[\alpha, \beta] \subset U$ which are complete and consistent and characterized by trapezoid membership functions with $d^0 = a^1 = b^1$, $c^M = d^M = a^{M+1}$. Define $\overline{x}^l$ as Theorem 1.

2) Construct M fuzzy IF-THEN rules for the LBFS as (1). Set the consequent part parameters to

$$\overline{y}^l = g(\overline{x}^l) \tag{4}$$

for all $1 \leq l \leq M$. $\overline{y}^l$ satisfies Theorem 1 because $g(\overline{x}^l) \geq h(\overline{x}^l)$.

3) Through the previous step, we can construct the $f(x)$ with the form of (2). Since it satisfies Theorem 1, it is the LBFS.

Now we prove that the constructed LBFS can serve as a universal approximator to any continuously differentiable and linearly bounded function.

Theorem 2: Let $f(x)$ be the constructed LBFS and $g(x)$ be the target function bounded by a linear function $h(x)$. If $g(x)$ is continuously differentiable on $U = [\alpha, \beta] \subset \mathbb{R}$ and satisfies

(4), then

$$\|f(x) - g(x)\|_\infty \leq \left\| \frac{dg(x)}{dx} \right\|_\infty \cdot \overline{x}^*,$$

where $\|d(x)\|_\infty = \sup_{x \in U} |d(x)|$, and $\overline{x}^* = \max_{0 \leq l \leq M} |\overline{x}^{l+1} - \overline{x}^l|$.

Proof: See Appendix B. ■

Theorem 2 shows that the constructed LBFS can approximate any continuously differentiable and linearly bounded function on a compact set U within a required error bound ϵ. But, in computing the required accuracy, we need to know the bound of the target function derivative with respect to its input.

IV. Simulations

We give an example that shows how to design an LBFS based on Theorem 1 and 2. As for a target function, we select

$$g(x) = \sin^2(\pi x) + x + 1 \tag{5}$$

where the bound is $h(x) = x + 1$ and $x \in U = [-1, 1]$. Given $\epsilon = 1.00$ and the constraint that $f(x) \geq h(x)$, we design an LBFS that approximates (5). Since $\left\| \frac{dg(x)}{dx} \right\|_\infty$ is calculated on a defined compact set U as

$$\left\| \frac{dg(x)}{dx} \right\|_\infty = 4.14,$$

$\overline{x}^* = 0.24$ meets the requirement. To construct an LBFS, we use complete and consistent fuzzy sets characterized by 10 trapezoid membership functions. Fig. 1 shows that the output of LBFS satisfies the linear constraint $h(x)$.

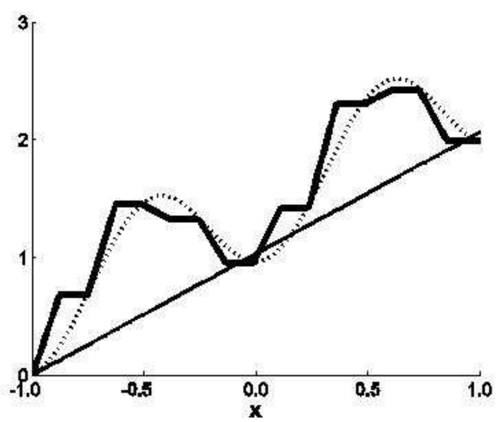

Fig. 1. Simulation for $\epsilon = 1.00$ (M = 10). The thick solid line is the output of LBFS; the dotted line is $g(x)$; the thin solid line is the linear bound $h(x)$

However, it has a jagged shape due to the wide flat cores of the trapezoid membership functions. To approximate $g(x)$ with a smooth LBFS, we reduce ϵ by placing more fuzzy sets along the x axis. For example, for a given $\epsilon = 0.50$, $\overline{x}^* = 0.11$ (M=20) meets the requirement (Fig. 2). More results that compare the error with rule number (M) is presented in Fig. 3.

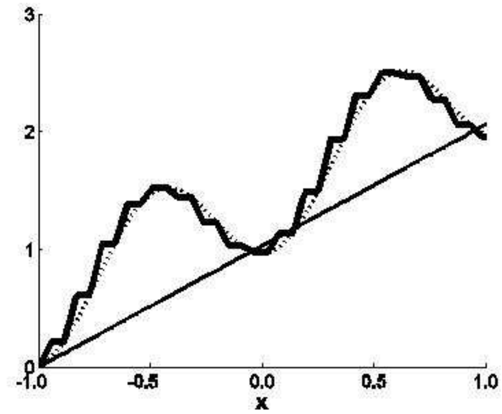

Fig. 2. Simulation for $\epsilon = 0.50$ (M = 20). Line specs are same to those of Fig. 1

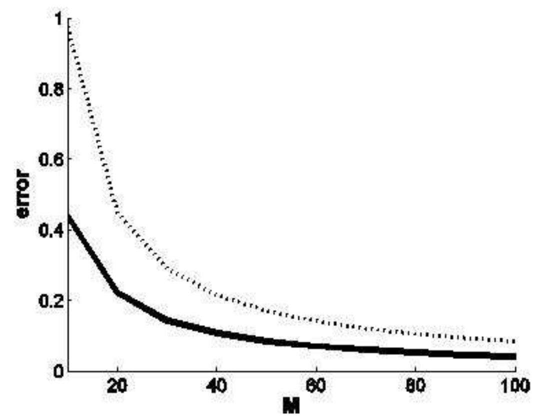

Fig. 3. Change in $\|f - g\|_\infty$ over M. The dotted line is the theoretical maximum error calculated by using Theorem 2; the solid line is the actual error calculated by using Matlab

V. Conclusion

In this paper, we present sufficient conditions for linearly bounded fuzzy system. When the fuzzy system is formed with the complete and consistent fuzzy set with the trapezoid membership functions, the derived condition is simply the constraints on the consequent part parameters. Then, we introduced one way of deriving an LBFS and showed that the developed LBFS can approximate any continuously differentiable and linearly bounded function on a compact set to any degree of accuracy. The validity of the LBFS has been demonstrated through simulation examples.

APPENDIX

A. Proof of Theorem 1

Since $f(x)$ is formed with the complete and consistent fuzzy sets that are characterized by the trapezoid membership functions, it can be described with at most two membership functions at each point $x \in U$. In fact, as shown in Fig. 4, only one membership function is nonzero in the intervals $[d^{l-1}, a^{l+1}]$ and two membership functions are nonzero in the intervals (a^{l+1}, d^1) for all $1 \leq l \leq M$. When $x \in [d^{l-1}, a^{l+1}]$, only $\mu^l(x) > 0$ and

$$\begin{aligned} f(x) =& \frac{\overline{y}^l \mu^l(x)}{\mu^l(x)} = \overline{y}^l \\ \geq& h(\overline{x}^l) \end{aligned}$$

Since $h(x)$ is a liner function, it has maximum at $x = a^{l+1}$ if $\pi_1 \geq 0$ or at $x = d^{l-1}$ if $\pi_1 < 0$. Therefore,

$$h(\overline{x}^l) = \max_{x \in [d^{l-1}, a^{l+1}]} h(x).$$

When $x \in (a^{l+1}, d^l)$, only $\mu^l(x)$ and $\mu^{l+1}(x)$ are positive. In these intervals, we need to show that

$$\begin{aligned} & f(x) - h(x) \\ \geq& \frac{\mu^l(x)h(\overline{x}^l) + \mu^{l+1}(x)h(\overline{x}^{l+1})}{\mu^l(x) + \mu^{l+1}(x)} - h(x) \\ =& \frac{\mu^l(x)(h(\overline{x}^l) - h(x)) + \mu^{l+1}(x)(h(\overline{x}^{l+1}) - h(x))}{\mu^l(x) + \mu^{l+1}(x)} \\ =& \pi_1 \frac{\mu^l(x)(\overline{x}^l - x) + \mu^{l+1}(x)(\overline{x}^{l+1} - x)}{\mu^l(x) + \mu^{l+1}(x)} \geq 0. \end{aligned}$$

Since $\mu^l(x) + \mu^{l+1}(x) > 0$, it is equivalent to show that

$$\pi_1 \left[\mu^l(x)(\overline{x}^l - x) + \mu^{l+1}(x)(\overline{x}^{l+1} - x) \right] \geq 0$$

If $\pi_1 > 0$, then $\overline{x}^l = a^{l+1}$ and

$$\begin{aligned} & \pi_1 \left[\mu^l(x)(a^{l+1} - x) + \mu^{l+1}(x)(a^{l+2} - x) \right] \\ \geq& \pi_1 \left[(1 - \mu^{l+1}(x))(a^{l+1} - x) + \mu^{l+1}(x)(b^{l+1} - x) \right] \\ =& \pi_1 \left[a^{l+1} - x + \mu^{l+1}(x)(b^{l+1} - a^{l+1}) \right] \\ =& \pi_1 \left(a^{l+1} - x + x - a^{l+1} \right) = 0 \end{aligned} \tag{6}$$

for all $1 \leq l \leq M$ because $a^{l+2} \geq b^{l+1}$ and

$$\begin{aligned} \mu^l(x) + \mu^{l+1}(x) =& \frac{d^l - x}{d^l - c^l} + \frac{x - a^{l+1}}{b^{l+1} - a^{l+1}} \\ \leq& \frac{b^{l+1} - x}{b^{l+1} - c^l} + \frac{x - c^l}{b^{l+1} - c^l} = 1. \end{aligned}$$

If $\pi_1 \leq 0$, then $\overline{x}^l = d^{l-1}$ and

$$\begin{aligned} & -\pi_1 \left[\mu^l(x)(x - d^{l-1}) + \mu^{l+1}(x)(x - d^l) \right] \\ \geq& -\pi_1 \left[\mu^l(x)(x - c^l) + (1 - \mu^l(x))(x - d^l) \right] \\ =& -\pi_1 \left[\mu^l(x)(d^l - c^l) + x - d^l \right] \\ =& -\pi_1 \left(d^l - x + x - d^l \right) = 0 \end{aligned} \tag{7}$$

because $d^{l-1} \leq c^l$ and $\mu^l(x) + \mu^{l+1} \leq 1$. Combining (6) and (7), we can show that $f(x) \geq h(x)$ in $x \in (a^{l+1}, d^l)$ for all $1 \leq l \leq M$.

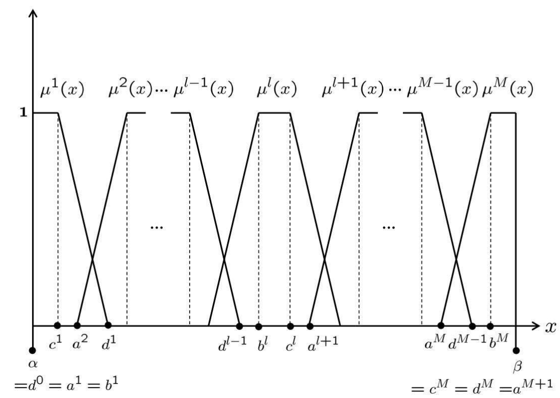

Fig. 4. Complete and consistent fuzzy sets characterized by the trapezoid membership functions

B. Proof of Theorem 2

Since the fuzzy sets used in $f(x)$ are complete and consistent, at most two membership functions $\mu^l(x)$ and $\mu^{l+1}(x)$ are nonzero for all $1 \leq l \leq M - 1$. Hence, the constructed LBFS in the form of (2) can be simplified to

$$\begin{aligned} f(x) =& \frac{\sum_{l=1}^M \overline{y}^l \mu^l(x)}{\sum_{l=1}^M \mu^i(x)} \\ =& \sum_{l=k}^{k+1} \left[\frac{\mu^l(x)}{\sum_{l=k}^{k+1} \mu^l(x)} \right] \overline{y}^l \end{aligned}$$

and, by using (4), we have

$$f(x) = \sum_{l=k}^{k+1} \left[\frac{\mu^l(x)}{\sum_{l=k}^{k+1} \mu^l(x)} \right] g(\overline{x}^l).$$

Since

$$\sum_{l=k}^{k+1}\left[\frac{\mu^l(x)}{\sum_{l=k}^{k+1}\mu^l(x)}\right]=1,$$

we have

$$\begin{aligned}|g(x)-f(x)| &\leq \sum_{l=k}^{k+1}\left[\frac{\mu^l(x)}{\sum_{l=k}^{k+1}\mu^l(x)}\right]|g(x)-g(\overline{x}^l)| \\ &\leq \max_{l=k,k+1}|g(x)-g(\overline{x}^l)|.\end{aligned}$$

From the Mean Value Theorem [9], we can derive the following inequality

$$|g(x)-f(x)| \leq \max_{l=k,k+1}\left\|\frac{dg(x)}{dx}\right\|_\infty |x-\overline{x}^l|. \tag{8}$$

Since $x \in [\overline{x}^k, \overline{x}^{k+1}]$, we know that $|x-\overline{x}^l| \leq |\overline{x}^{k+1}-\overline{x}^k|$ for $l=k,k+1$. Hence, (8) becomes

$$|g(x)-f(x)| \leq \left\|\frac{dg(x)}{dx}\right\|_\infty |\overline{x}^{k+1}-\overline{x}^k| \tag{9}$$

from which, we have

$$\begin{aligned}\|g(x)-f(x)\| &= \sup_{x\in U}|g(x)-f(x)| \\ &\leq \left\|\frac{dg(x)}{dx}\right\|_\infty \max_{0\leq k\leq M}|\overline{x}^{k+1}-\overline{x}^k| \\ &= \left\|\frac{dg(x)}{dx}\right\|_\infty \cdot \overline{x}^*, \end{aligned} \tag{10}$$

where $\overline{x}^* = \max_{0\leq l\leq M}|\overline{x}^{l+1}-\overline{x}^l|$. Since $g(x)$ is continuously differentiable for all $x \in U$, $\left\|\frac{dg(x)}{dx}\right\|_\infty$ is finite in (10) and, given any $\epsilon > 0$, we can choose $\overline{x}^*$ small enough to satisfy $\|g(x)-f(x)\|_\infty < \epsilon$.

References

[1] Hassan K. Khalil, Nonlinear Systems, Prentice-Hall, 3rd ed. 2002.

[2] L. X. Wang, A Course in Fuzzy Systems and Control, Prentice-Hall, 1997.

[3] L. X. Wang and J. M. Mendel, "Fuzzy basis functions, universal approximation, and orthogonal least square learning," IEEE Trans. Neural Networks, vol. 3, no.5, pp. 807–814, 1992.

[4] L.X. Wang, Fuzzy systems are universal approximators, in Proc. IEEE Int. Conf. Fuzzy Syst., San Diego, CA, Mar. 1992, pp. 1163-1170.

[5] J. L. Castro and M. Delgado, Fuzzy systems with defuzzification are universal approximators, IEEE Trans. Syst. Man Cybern. B, Cybern., vol. 26, no. 1, pp. 149-152, Feb. 1996.

[6] B. Kosko, Fuzzy systems as universal approximators, in Proc. IEEE Int. Conf. Fuzzy Syst., San Diego, CA, Mar. 1992, pp.

[7] F. L. Lewis, S.-Q. Zhu, and K. Liu, Function approximation by fuzzy systems, in Proc. 1995 American Control Conf., Seattle, WA, Jun. 1995, pp. 3760-3764.

[8] H. T. Nguyen, V. Kreinovich, and O. Sirisaengtaksin, Fuzzy control as a universal control tool, Fuzzy Sets Syst., vol. 80, no. 1, pp. 71-86, May 1996.

[9] Walter Rudin, Principles of Mathematical Analysis, McGrawHill, 3rd ed. 1976.