MONOTONIC FUZZY SYSTEMS AS UNIVERSE AL APPROXIMATORS FOR MONOTONIC FUNCTIONS

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ABSTRACT—In this paper, we propose a constructive method to develop a fuzzy system having a monotonic input–output relationship and prove that the developed fuzzy system can approximate any continuously differentiable monotonic function with any desired degree of accuracy. The fuzzy system is constructed with complete and consistent input membership functions and imposes special parametric constraints on the consequent part of the fuzzy rules. The monotonicity property and approximation capability of the developed fuzzy system are demonstrated using numerical examples.

Key Words: Fuzzy constraint satisfaction, fuzzy systems, monotonic fuzzy systems, monotonicity constraints, universal approximation

1. INTRODUCTION

Input-output monotonicity is a very common property of physical systems [1]–[5]. When approximating such monotonic systems with a fuzzy model, imposing a monotonicity constraint on the fuzzy model is a useful approach to reduce training effort and minimize approximation error [6], [7]. As an example, consider the automated current control of magnet cranes in the steel plate yard [8]. Traditionally, this operation has been managed by the operator who exerts the proper amount of current on the magnet to lift the right number of plates and move them over to the destination. This current control procedure is often automated by using a fuzzy model whose parameters are determined by using the available input–output data: the amount of current supplied by the crane operator (input) and the number of plates to lift (output). In this case, we do not know the exact function between its input and output, but we do know clearly that the input and output have the monotonic relationship: more current is required to lift more plates.

In this study, a fuzzy system whose output is monotonic with respect to its input is defined as a monotonic fuzzy system (MFS). Despite its significance in approximating many physical systems under monotonicity constraints, literature concerning MFS is quite sparse. Wu et al. [1], [2] introduced a fuzzy controller with a mean-of-inversion defuzzification technique for control of image compression systems. In this system, the average gray scale error between the original
image and the compressed image was found to be monotonic with respect to the input parameter quality, which is used to design a fuzzy controller that leads to the desired performance. Lindskog et al. [3] proposed a fuzzy model that ensures input–output monotonicity and used it to identify a dynamic system whose steady state output is monotonic with respect to its constant input. Zhao et al. [4] presented the concept of a monotonic rule base and derived a sufficient condition for the P or PD-type fuzzy control algorithms to be monotonic. V. S. Kouikoglou and Y. A. Phillis [5] presented sufficient conditions on the parameters of hierarchical fuzzy systems to be monotonic.

More recently, Won et al. [9] identified sufficient conditions under which a fuzzy system becomes monotonic. Derived from the first derivative of fuzzy output equation, Won’s conditions are general enough to be applicable to any fuzzy system whose output is differentiable with respect to an input. Two conditions apply. One is a geometrical constraint on the input membership functions and the other is a parametric constraint on the consequent part. The latter constraint has been successfully used to approximate MFS using fuzzy identification techniques such as evolutionary algorithms and least-squares (LS) learning [6], [7].

Regarding Won’s conditions, an important question arises: Can an MFS that satisfies Won’s conditions approximate all monotonic functions? In fact, it has been proven that there exists a general fuzzy system that can approximate any continuous or \( L_2 \) function to any specified precision [10]–[23]. It also has been proven that there exists a hierarchical fuzzy system that can approximate any continuous function to any specified precision [24]–[28]. However, it has not been proven whether an MFS can approximate all monotonic functions. This study shall answer this question by suggesting a constructive method to generate an MFS. Two examples are given to illustrate the monotonicity property and approximating capability of the MFS generated using the suggested method.

This paper is organized as follows: In Section 2, we derive sufficient conditions for a fuzzy system to be monotonic. In Section 3, we show in a constructive manner that the MFS developed in Section 2 can serve as an approximator for any continuously differentiable monotonic function. In Section 4, we give simulation examples and in Section 5, we draw conclusions.

2. MONOTONIC FUZZY SYSTEM

An MFS is a fuzzy system whose output is monotonic with respect to its input. The system could be either monotonically increasing or monotonically decreasing but, without loss of generality, we consider only the monotonically increasing case in this paper.

**Definition 1 (Vector Ordering)** Given \( x^1 = (x_1^1, \cdots, x_n^1)^T \) and \( x^2 = (x_1^2, \cdots, x_n^2)^T \), \( x^1 < x^2 \) (\( x^1 \leq x^2 \)) if and only if \( x_k^1 < x_k^2 \) (\( x_k^1 \leq x_k^2 \)) for all \( 1 \leq k \leq n \). Note that \( x^2 > x^1 \) (\( x^2 \geq x^1 \)) if and only if \( x^1 < x^2 \) (\( x^1 \leq x^2 \)).

**Definition 2 (Monotonic Function)** Let \( x = (x_1, \cdots, x_n)^T \in U = [\alpha_1, \beta_1] \times \cdots \times [\alpha_n, \beta_n] \subseteq \mathbb{R}^n \) be an input vector of the function \( y = f(x) \in V \subseteq \mathbb{R} \). Then, \( f: U \mapsto V \) is said to be **monotonically increasing** if \( f(x^1) \leq f(x^2) \) whenever \( x^1 \leq x^2 \) where \( x^1 = (x_1^1, \cdots, x_n^1)^T \) and \( x^2 = (x_1^2, \cdots, x_n^2)^T \).

Assuming that a given function is continuously differentiable, it is monotonically increasing if all of its partial derivatives are nonnegative [9].
Consider an \( n \)-input single-output fuzzy system, where \( M_j \) fuzzy sets are assigned to each input variable \( 1 \leq j \leq n \). The IF-THEN rules of the fuzzy system are of the following form:

\[
R^{(l_1, \cdots, l_n)}: \text{IF } x_1 \text{ is } A_1^{l_1} \text{ and } \cdots \text{ and } x_n \text{ is } A_n^{l_n} \text{ THEN } y \text{ is } B^{(l_1, \cdots, l_n)}, \tag{1}
\]

where \( A_j^{l_j}, B^{(l_1, \cdots, l_n)} \) are fuzzy sets and \( 1 \leq l_j \leq M_j \) is an index for the input fuzzy set associated with input \( x_j \) for \( 1 \leq j \leq n \). If full grid partitions are used for each of the input variables, then the total number of rules is \( \prod_{j=1}^{n} M_j \). If we use singleton fuzzifier, center average defuzzifier and product inference engine, the output of the fuzzy system that we analyze in this paper can be formulated as

\[
y = F(x) = \frac{\sum_{l_1=1}^{M_1} \cdots \sum_{l_n=1}^{M_n} \prod_{j=1}^{n} \mu_{A_j^{l_j}}(x_j) \prod_{j=1}^{n} \mu_{B_j^{l_j}}(x_j)}{\sum_{l_1=1}^{M_1} \cdots \sum_{l_n=1}^{M_n} \prod_{j=1}^{n} \mu_{A_j^{l_j}}(x_j)}, \tag{2}
\]

where \( \mu_{A_j^{l_j}}(x_j) \) is the membership function of the fuzzy set \( A_j^{l_j} \) and \( \overline{y}^{(l_1, \cdots, l_n)} \), what we call a consequent part parameter, is the center of \( B^{(l_1, \cdots, l_n)} \).

To characterize the fuzzy sets and membership functions used for (1) and (2), we introduce the following definitions [10], [22].

**Definition 3 (Pseudo-Trapezoidal (PT) Membership Function)** For \( a, b, c, d, \) and \( U = [\alpha, \beta] \) such that \( \alpha \leq a \leq b \leq c \leq d \leq \beta \), the PT membership function over \( U \) is given by

\[
\mu(x; a, b, c, d) = \begin{cases} 
I(x), & x \in [a, b) \\
1, & x \in [b, c] \\
D(x), & x \in (c, d] \\
0, & \text{otherwise}
\end{cases}
\]

where \( I(x) \) and \( D(x) \) are monotonically increasing and decreasing functions of \( x \), respectively, such that \( 0 \leq I(x) \leq 1 \) and \( 0 \leq D(x) \leq 1 \).

**Definition 4 (Completeness)** Fuzzy sets \( A_1, \cdots, A^n \) in \( U \subset \mathbb{R} \) are said to be complete on \( U \) if for any \( x \in U \), a fuzzy set \( A^k \) exists such that \( \mu_{A^k}(x) > 0 \).

**Definition 5 (Consistency)** Fuzzy sets \( A_1, \cdots, A^n \) in \( U \subset \mathbb{R} \) are said to be consistent on \( U \) if whenever \( \mu_{A_j}(x) = 1 \) for some \( x \in U \), then \( \mu_{A_k}(x) = 0 \) for all \( k \neq l \).

**Definition 6 (High Set)** The high set of a fuzzy set \( A^k \) in \( U \subset \mathbb{R} \) is defined by

\[
\text{high}(A^k) = \left\{ x \in U \mid \mu_{A^k}(x) = \sup_{x' \in U} \mu_{A^k}(x') \right\}.
\]
Definition 7 (Order between Fuzzy Sets) For two fuzzy sets $A^k$ and $A^l$ in $U \subseteq \mathbb{R}$ we define $A^k < A^l$ if $x' < x$ for any $x' \in \text{high}(A^k)$ and $x \in \text{high}(A^l)$.

Suppose that the fuzzy system (2) uses complete and consistent input fuzzy sets characterized by PT membership functions. Based on the results of [9], we can now derive sufficient conditions on the parameters for this multi-input fuzzy system to be monotonic with respect to $x_j$ for all $1 \leq j \leq n$.

**Theorem 1:** The fuzzy system $F(x)$ as given in (2) with complete and consistent input fuzzy sets characterized by PT membership functions is monotonically increasing with respect to $x_j$ in $U_j$ if

$$\bar{y}(l_{i_1} \cdots l_{j-1}, p_{j+1}, \cdots l_n) \leq \bar{y}(l_{i_1} \cdots l_{j-1}, q_{j+1}, \cdots l_n)$$

(3)

for all possible combinations of fuzzy sets associated with the plane $\left(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n\right)^T$ where $1 \leq p \leq q \leq M_j$.

**proof:** See the appendix. Q.E.D.

Note here that PT membership functions include most of popular membership functions such as triangular membership functions, trapezoidal membership functions and bell-shaped membership functions. One popular membership function that complete and consistent PT membership functions do not include is the Gaussian membership function. In fact, [9] shows that we can construct an MFS with Gaussian membership functions but in this case the involved Gaussian membership functions must be with equal variance. This constraint is overly restrictive in practical applications because, as in control system applications, we could use them with large variances when the actual output is far away from the desired set point but we may have to use those with small variances (for accurate control) when the actual output gets close to the desired set point. It is true that we could use the same small variances for all Gaussian membership functions to build an MFS but it may require unacceptably large number of rules and impractical. Therefore we restrict our attention to complete and consistent PT membership functions in this paper where we can use different widths or variances for different PT membership functions.

### 3. UNIVERSAL APPROXIMATOR OF MONOTONIC FUNCTION

In this section, we prove constructively that the developed MFS can approximate any continuously differentiable monotonic function on a compact set to any required accuracy. Let $g(x): U \subseteq \mathbb{R}^n \rightarrow V \subseteq \mathbb{R}$ be the target monotonic function of the MFS satisfying $g(x^1) \leq g(x^2)$ whenever $x^1 \leq x^2$. We assume that $g(x)$ is unknown but that the values of $g(x)$ are available for finite input points $x$ in $U$. We construct an MFS approximating $g(x)$ as follows:

1) Define $M_j$ fuzzy sets $A_{j_1}^1, \cdots, A_{j_1}^{M_j}$ in $[\alpha_j, \beta_j] \subseteq U_j$, which are complete and consistent, are characterized by PT membership functions with $a_j^1 = b_j^1 = \alpha_j$ and
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Let \( c_j^{M_j} = d_j^{M_j} = \beta_j \), and are arranged as \( A_1^j < \cdots < A_n^{M_j} \) for \( 1 \leq j \leq n \). Define \( \overline{x}_j = \alpha_j \), \( \overline{x}_j^{M_j} = \beta_j \) and \( \overline{x}_j^{k_j} = (h_j^{k_j} + c_j^{k_j})/2 \) for \( 2 \leq k_j \leq M_j - 1 \) and \( 1 \leq j \leq n \).

2) Construct \( M = \prod_{j=1}^n M_j \) fuzzy IF-THEN rules for the multi-input MFS as (1). Set the consequent part parameters to

\[
\overline{y}(h_1, \cdots, h_n) = g\left( \overline{x}(h_1, \cdots, h_n) \right)
\]

where \( \overline{x}(h_1, \cdots, h_n) = (\overline{x}_1 \cdots, \overline{x}_n)^T \). Because \( g(x) \) is monotonic,

\[
\overline{y}(h_1, \cdots, h_n, p_{d_1}, \cdots, d_n) \leq \overline{y}(h_1, \cdots, h_n, p_{d_1}, \cdots, d_n)
\]

whenever \( \overline{x}(h_1, \cdots, h_n, p_{d_1}, \cdots, d_n) \leq \overline{x}(h_1, \cdots, h_n, p_{d_1}, \cdots, d_n) \) for all combinations of \( (x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)^T \) and \( 1 \leq p \leq q \leq M_j \).

Now, we prove that the constructed MFS can serve as an approximator to any continuously differentiable monotonic function.

**Theorem 2:** Let \( F(x) \) be the constructed MFS and \( g(x) \) be the target monotonic function. If \( g(x) \) is continuously differentiable on \( U = [\alpha_1, \beta_1] \times \cdots \times [\alpha_n, \beta_n] \subset \mathbb{R}^n \), then

\[
\|g - F\|_{\infty} \leq \left\| \frac{\partial g}{\partial x_1} \right\|_{\infty} h_1 + \cdots + \left\| \frac{\partial g}{\partial x_n} \right\|_{\infty} h_n
\]

where \( \|d(x)\|_{\infty} = \sup_{x \in U} |d(x)| \) for any function \( d(x) \) and \( h_j = \max_{1 \leq k_j \leq M_j - 1} |\overline{x}_j^{k_j+1} - \overline{x}_j^{k_j}| \) for \( 1 \leq j \leq n \).

**Proof:** See the appendix. Q.E.D.

In (5), \( \left\| \frac{\partial g}{\partial x_1} \right\|_{\infty}, \cdots, \left\| \frac{\partial g}{\partial x_n} \right\|_{\infty} \) are finite numbers and we can choose the value of \( h_1, \cdots, h_n \) to satisfy any given error \( \varepsilon > 0 \). That is, \( \|g(x) - F(x)\|_{\infty} \leq \varepsilon \). Therefore, Theorem 2 shows that the constructed MFS can approximate any continuously differentiable monotonic function on a compact set \( U \) to a required error bound \( \varepsilon \). But, in computing the required accuracy, we need to know the bounds of the derivatives of the target function with respect to its inputs.

Note here that we can use [23] to prove that there exists an MFS that approximates a given continuously differentiable monotonic function. But in this case the MFS must be constructed with Gaussian membership functions with equal variance, which is overly restrictive as mentioned in Section 2. And it is a mathematical approach that proves only the existence of an MFS that approximates any continuously differentiable monotonic function. On the other hand, Theorem 2 applies to an MFS built with PT membership functions with flexible widths. In addition, the proof of Theorem 2 is constructive and actually shows how to construct an MFS with PT membership functions.
4. NUMERICAL EXAMPLES

In this section, we present two examples illustrating the application of the developed MFS as an approximator for continuously differentiable monotonic function. In the first example, a simple two-dimensional target function was approximated by the MFS. In the second example, the dynamic behavior of a gas furnace system was characterized using a batch LS approach based on the MFS model.

4.1 Monotonic Function Approximation

In this example, a two-input monotonic target function was selected as:

\[ g(x) = \frac{1}{4} \left( \tanh(0.4x_1) + \tanh(0.4x_2) \right), \]

where \( (x_1, x_2) \in \mathbb{R}^2 \) (Figure 1).

Figure 1. Plot of the target function.

We assume that \( g(x) \) is unknown but that \( \| \frac{\partial g}{\partial x_i} \|_\infty = \| \frac{\partial g}{\partial x_n} \|_\infty = 0.1 \) on the defined compact set \( U \) and the values of \( g(x) \) at \( x = (x_1^i, x_2^i) \) are known.
Given \( \varepsilon = 0.4 \), we designed a two-input single-output MFS that approximates (6). \( h_1 = h_2 = 2 \) could achieve approximation error no greater than \( \varepsilon = 0.4 \). To construct the MFS, we used trapezoidal membership functions

\[
\mu_{A_j^i}(x) = \mu_{A_j^i}(x; -5, -5, \bar{x}_j^i + 0.4h_j, \bar{x}_j^i + 0.6h_j),
\]

\[
\mu_{A_j^p}(x) = \mu_{A_j^p}(x; \bar{x}_j^p + 0.6h_j, \bar{x}_j^p + 0.4h_j, \bar{x}_j^p + 0.4h_j, \bar{x}_j^p + 0.6h_j),
\]

\[
\mu_{A_j^s}(x) = \mu_{A_j^s}(x; \bar{x}_j^s - 0.6h_j, \bar{x}_j^s - 0.4h_j, 5, 5),
\]

where \( \bar{x}_j^p = 2p - 7 \) for \( 1 \leq p \leq 6 \) and \( 1 \leq j \leq 2 \). This arrangement of trapezoidal membership functions is complete and consistent (Figure 2). The consequent part parameter \( \mu_g^{(h_1, h_2)} \) was set to \( g(\bar{x}_1^h, \bar{x}_2^h) \) to satisfy (3).

![Figure 2. Arrangement of trapezoidal membership functions for MFS when \( \varepsilon = 0.4 \) (\( j = 1, 2 \)).](image)

We obtained the output equation of the fuzzy system as:

\[
F(x) = \frac{\sum_{h_1=1}^{6} \sum_{h_2=1}^{6} g(\bar{x}_1^{h_1}, \bar{x}_2^{h_2}) \prod_{j=1}^{2} \mu_{A_j^i}(x_j)}{\sum_{h_1=1}^{6} \sum_{h_2=1}^{6} \prod_{j=1}^{2} \mu_{A_j^i}(x_j)}
\]

(Figure 3). As expected, \( F(x) \) was monotonic with respect to \( x_1 \) and \( x_2 \), and approximated \( g(x) \) within the required error bound. However, it had a jagged shape due to the wide flat cores of the trapezoidal membership functions. To approximate \( g(x) \) with a smoother MFS, we might reduce \( \varepsilon \) and place more fuzzy sets along the \( x_1 \) and \( x_2 \) axes. For example, if \( \varepsilon = 0.2 \), we could approximate \( g(x) \) with 11 fuzzy sets for each input space. Repeating the above procedure, we obtained another MFS whose surface is very close to \( g(x) \) (Figure 4).
4.2 Nonlinear System Identification

We also attempted to apply the fuzzy system satisfying (3) in Theorem 1 to Box-Jenkins data [29] obtained from a gas furnace system. The gas furnace system has a gas flow rate (cubic feet per minute) \( u \in [-2.716, 2.834] \) as a single input and \( CO_2 \) concentration (%) \( y \in [45.6, 60.5] \) as a single output. The \( CO_2 \) concentration at time \( t, y_t \), was simplified as a two-input nonlinear function of \( u_{t-4} \) and \( y_{t-1} \) which were generally accepted variables in the two-input case [30].

The exact formula of the nonlinear function is unknown, but it is certain that the relationship between \( y_t \) and \( u_{t-4} \) is monotonic and increasing. Because the relationship between \( y_t \) and \( y_{t-1} \) is not monotonic, the monotonicity constraint is imposed on the relationship between \( y_t \) and \( u_{t-4} \) only. To approximate the nonlinear function, we established a two-input fuzzy system whose input \( x_1 \) and \( x_2 \) respectively represent \( u_{t-4} \) and \( y_{t-1} \). We arranged membership functions for \( x_1 \) to become complete and consistent (Figure 5). Membership functions for \( x_2 \) were also distributed to cover the range of \( y_{t-1} \) (Figure 6). Then the output of the fuzzy system is formulated as:
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Figure 4. MFS plot for $\varepsilon = 0.2$.

Figure 5. Arrangement of membership functions for $x_1$ variable.

$$F(x) = \frac{\sum_{l_1=1}^{3} \sum_{l_2=1}^{6} \nu^{(l_1,l_2)} \prod_{j=1}^{2} \mu_{A_{ij}}(x_j)}{\sum_{l_1=1}^{3} \sum_{l_2=1}^{6} \prod_{j=1}^{2} \mu_{A_{ij}}(x_j)}.$$
To train and test this fuzzy model, we generated 292 triples of \((u_{t-4}, y_{t-1}, y_t)\) from Box-Jenkins data. Only \(\tilde{y}^{(l_2)}\) was trained to learn the dynamics of the system using a batch LS approach. We compared two LS approaches. One was the conventional LS method and the other was the constrained LS method subject to the following condition derived from (3):

\[
45.6 \leq \tilde{y}^{(1, l_2)} \leq \tilde{y}^{(2, l_2)} \leq \tilde{y}^{(3, l_2)} \leq 60.5
\]

for all \(1 \leq l_2 \leq 6\). The fuzzy model trained by the latter LS method maintains the monotonic input–output relationship. For both LS methods, we used the first half of data triples for training and the second half for test.

In the training phase, the root mean square errors (RMSE) were identical for the two fuzzy models. In the testing phase, however, the RMSE of the fuzzy model trained by the constrained LS method was 22% less than that of the conventional LS method (0.5498 vs. 0.7075). The results are given in Figure 7 and Figure 8. This shows that the fuzzy model trained under the monotonicity constraint yielded more precise identification results than the model trained without this constraint.

5. CONCLUSION

In this paper, we proved that a fuzzy system developed in a special constructive manner guarantees a monotonic input–output relationship and can approximate any continuously differentiable monotonic function on a compact set to any desired degree of accuracy. To the authors’ knowledge, this is the first result to prove the universal approximation of a constrained fuzzy system for monotonic functions. This work provided a constructive proof of the universal approximator for a special type of fuzzy system and presents several interesting challenges for future research. In this study, we considered only PT membership functions, which include triangular membership functions, trapezoidal membership functions and bell-shaped membership functions, in the constructive method, but the method cannot be generalized to some membership functions such as Gaussian membership functions. We are currently working to prove that arbitrary fuzzy systems that satisfy the suggested conditions in Theorem 1 are universal approximators of any continuously differentiable monotonic functions. We will also attempt to derive analytical conditions on fuzzy parameters under which the fuzzy system has a convex input–output relationship.
Figure 7. Comparison of desired output with the predicted output for the conventional fuzzy model (solid line: desired output, dashdot line: prediction output).

**APPENDIX**

**Proof of Theorem 1:**

Because a multi-input function is monotonic whenever it is monotonic for each of its inputs, (2) can be transformed to

\[
y = F(x) = \frac{\sum_{k=1}^{M_j} \sum_{l_j=1}^{M_j} \sum_{j=1}^{n} y^{(k,l_j)}(x_j) \prod_{i=1,i\neq j}^{n} \mu_{x^i}(x_i)}{\sum_{k=1}^{M_j} \sum_{l_j=1}^{M_j} \sum_{j=1}^{n} \mu_{x^j}(x_j) \prod_{i=1,i\neq j}^{n} \mu_{x^i}(x_i)},
\]

where the index \( k \) is for all possible combinations of fuzzy sets associated with the plane \((x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)^T\) where \( 1 \leq k \leq M^t = \prod_{i=1,i\neq j}^{n} M_i \), and the index \( l_j \) is for the fuzzy sets of \( x_j \) where \( 1 \leq l_j \leq M_j \).
Figure 8. Comparison of desired output with the predicted output for the developed MFS (solid line: desired output, dashdot line: prediction output).

Note here that the involved PT membership functions are continuous in $U_j$ but may not be differentiable at some finite points in $U_j$. Let $\Gamma_j = \{e_j^1, \cdots, e_j^m\}$ be the finite set of points in $U_j = [\alpha_j, \beta_j]$ where the membership functions are not differentiable. Differentiating (7) with respect to $x_j$ in $\overline{U}_j = U_j - \Gamma_j$ yields
\[
\frac{dy}{dx_j} = \frac{1}{v^2} \left\{ \frac{M'}{m} \sum_{k=1}^{M'} \sum_{p=1}^{M_j} y^{(k,p)}(x_j) \frac{d\mu_{\alpha^p_j}(x_j)}{dx_j} \chi^k \sum_{s=1}^{M'} \sum_{q=1}^{M_j} \mu_{\alpha^q_j}(x_j) \chi^s \right\} \\
- \frac{1}{v^2} \left\{ \frac{M'}{m} \sum_{k=1}^{M'} \sum_{s=1}^{M_j} \sum_{q=1}^{M_j} \left( \overline{y}^{(k,p)} - \overline{y}^{(k,q)} \right) \frac{d\mu_{\alpha^p_j}(x_j)}{dx_j} \mu_{\alpha^q_j}(x_j) \chi^k \chi^s \right\} \\
= \frac{1}{v^2} \left\{ \frac{M'}{m} \sum_{k=1}^{M'} \sum_{s=1}^{M_j} \sum_{q=1}^{M_j} \left( \overline{y}^{(k,p)} - \overline{y}^{(k,q)} \right) \left( \frac{d\mu_{\alpha^p_j}(x_j)}{dx_j} \mu_{\alpha^q_j}(x_j) \frac{d\mu_{\alpha^p_j}(x_j)}{dx_j} \right) \right\} \\
- \frac{1}{v^2} \left\{ \frac{M'}{m} \sum_{k=1}^{M'} \sum_{s=1}^{M_j} \sum_{q=1}^{M_j} \left( \overline{y}^{(k,p)} - \overline{y}^{(k,q)} \right) \left( \frac{d\mu_{\alpha^p_j}(x_j)}{dx_j} \mu_{\alpha^q_j}(x_j) \frac{d\mu_{\alpha^p_j}(x_j)}{dx_j} \right) \right\} \\
\times \left\{ \frac{d\mu_{\alpha^p_j}(x_j)}{dx_j} \mu_{\alpha^q_j}(x_j) \frac{d\mu_{\alpha^p_j}(x_j)}{dx_j} \right\},
\]

where \( v = \sum_{k=1}^{M'} \sum_{s=1}^{M_j} \mu_{\alpha^p_j}(x_j) \prod_{i=1, i \neq j}^{n} \mu_{\alpha^p_i}(x_i) \) and \( \chi^k = \prod_{i=1, i \neq j}^{n} \mu_{\alpha^p_i}(x_i) \) and \( \chi^s = \prod_{i=1, i \neq j}^{n} \mu_{\alpha^q_i}(x_i) \). It follows from (8) that \( \left( \frac{dy}{dx_j} \right) \geq 0 \) and that the fuzzy system is monotonically increasing with respect to \( x_j \) in \( \tilde{U}_j \) if \( \overline{y}^{(k,p)} \leq \overline{y}^{(k,q)} \) and

\[
\left( \frac{d\mu_{\alpha^p_j}(x_j)}{dx_j} \right) \mu_{\alpha^q_j}(x_j) - \mu_{\alpha^p_j}(x_j) \left( \frac{d\mu_{\alpha^p_j}(x_j)}{dx_j} \right) \leq 0
\]

are satisfied for all \( x_j \in \tilde{U}_j \) where \( 1 \leq k \leq M' \) and \( 1 \leq p \leq q \leq M_j \).

Figure 9. An example of two adjacent complete and consistent fuzzy sets with PT membership functions.
On the other hand, note that if the fuzzy sets are complete and consistent, then PT membership functions satisfy
\[
|\mu_{A_j}(x_j)| - |\mu_{A_j^{p+1}}(x_j)| = 0.
\]
In the overlapped region between \((c_j^p, d_j^p)\) of \(\mu_{A_j}(x_j)\) and \([a_j^{p+1}, b_j^{p+1})\) of \(\mu_{A_j^{p+1}}(x_j)\), because \(I(x_j)\) is monotonically increasing and \(D(x_j)\) is monotonically decreasing, then \(d\mu_{A_j^{p+1}}(x_j)/dx_j \leq 0\) and \(d\mu_{A_j}(x_j)/dx_j \geq 0\) for all \(x_j \in \bar{U}_j\). In addition, \(\mu_{A_j}(x_j) \geq 0\) and \(\mu_{A_j^{p+1}}(x_j) \geq 0\) for all \(x_j \in \bar{U}_j\).

Consequently,
\[
\frac{d\mu_{A_j^{p+1}}(x_j)}{dx_j} - \mu_{A_j^{p+1}}(x_j) - \mu_{A_j}(x_j) \cdot \frac{d\mu_{A_j}(x_j)}{dx_j} \leq 0,
\]
for all \(1 \leq p \leq M_j - 1\) and the fuzzy system \(F(x)\) becomes monotonically increasing with respect to \(x_j\) in \(\bar{U}_j\) if \(\bar{y}(k,p) = \bar{y}^{(l_1,\cdots,l_{j+1},p,l_{j+1},\cdots,l_n)} \leq \bar{y}^{(l_1,\cdots,l_{j+1},q,l_{j+1},\cdots,l_n)} = \bar{y}^{(k,q)}\) is satisfied for \(1 \leq p \leq q \leq M_j\).

Next, if we can show that \(F(x)\) is monotonically increasing with respect to \(x_j\) in \([e_j^i, e_j^{i+1}]\) for \(1 \leq i \leq m - 1\), then it becomes monotonically increasing with respect to \(x_j\) in \(U_j\) and the proof is complete.

To avoid notational complexity, we use the following notations in this paragraph:
\[
x_j^0 = (x_1, \cdots, x_j^0, \cdots, x_n)^T \quad \text{and} \quad x_j^i = (x_1, \cdots, e_j^i, \cdots, x_n)^T,
\]
where \(1 \leq j \leq n\) and \(1 \leq i \leq m\). Assume that \(F(x_j^i)\) is not the minimum of \(F(x)\) among \(x_j\) in \([e_j^i, e_j^{i+1}]\) and derive a contradiction. If \(F(x_j^i)\) is not the minimum among \(x_j\) in \([e_j^i, e_j^{i+1}]\), then there exists a \(x_j^0 \in (e_j^i, e_j^{i+1}]\) such that \(F(x_j^0) > F(x_j^i)\). Because \(F(x)\) is monotonically increasing with respect to \(x_j\) in \((e_j^i, e_j^{i+1}]\) and \(F(x_j^i) \leq F(x_j^0)\) for all \(x_j^0 \in (e_j^i, e_j^{i+1}]\), we cannot find a \(\delta > 0\) such that \(|x_j^i - x| < \delta\) implies \(|F(x_j^i) - F(x)| < |F(x_j^0) - F(x_j^i)| < \epsilon = (F(x_j^i) - F(x_j^0))/2\). This failure contradicts the premise that \(F(x)\) is continuous in \(U\). Therefore, \(F(x_j^i)\) must be the minimum of \(F(x)\) with respect to \(x_j\) in \([e_j^i, e_j^{i+1}]\). Using the same procedure, we can prove that \(F(x_j^{i+1})\) is the maximum of \(F(x)\) with respect to \(x_j\) in \([e_j^i, e_j^{i+1}]\). Therefore, \(F(x)\) is monotonically increasing with respect to \(x_j\) in \([e_j^i, e_j^{i+1}]\) for all
1 ≤ i ≤ n − 1 and hence in \( U_j \). Because this result is valid for all 1 ≤ j ≤ n, \( F(\mathbf{x}) \) is monotonically increasing in \( U \).

**Proof of Theorem 2:**

Because fuzzy sets used in \( F(\mathbf{x}) \) are complete and consistent, at most two PT membership functions, i.e., \( \mu_{A_{kj}}(x_j) \) and \( \mu_{A_{kj+1}}(x_j) \) are nonzero, where \( k_j \) and \( k_{j+1} \) denote the index of nonzero membership functions for \( j \) th (1 ≤ j ≤ n) input variable. Hence, the constructed MFS in the form of (2) can be simplified to

\[
F(\mathbf{x}) = \frac{\sum_{l_1=k_1}^{k_1+1} \cdots \sum_{l_n=k_n}^{k_n+1} \prod_{j=1}^{n} \mu_{A_{kj}}(x_j)}{\sum_{l_1=k_1}^{k_1+1} \cdots \sum_{l_n=k_n}^{k_n+1} \prod_{j=1}^{n} \mu_{A_{kj}}(x_j)}
\]

and using (4) yields

\[
F(\mathbf{x}) = \sum_{l_1=k_1}^{k_1+1} \cdots \sum_{l_n=k_n}^{k_n+1} \left[ \frac{\prod_{j=1}^{n} \mu_{A_{kj}}(x_j)}{\sum_{l_1=k_1}^{k_1+1} \cdots \sum_{l_n=k_n}^{k_n+1} \prod_{j=1}^{n} \mu_{A_{kj}}(x_j)} \right] \mathbf{g}\left(\mathbf{x}^{(l_1, \ldots, l_n)}\right).
\]

Because

\[
\sum_{l_1=k_1}^{k_1+1} \cdots \sum_{l_n=k_n}^{k_n+1} \left[ \frac{\prod_{j=1}^{n} \mu_{A_{kj}}(x_j)}{\sum_{l_1=k_1}^{k_1+1} \cdots \sum_{l_n=k_n}^{k_n+1} \prod_{j=1}^{n} \mu_{A_{kj}}(x_j)} \right] = 1,
\]

we have
\[ g(x) = \sum_{l_i = k_i}^{k_i+1} \cdots \sum_{l_n = k_n}^{k_n+1} \left[ \prod_{j=1}^{n} \mu_{A_j}^{x_j}(x_j) \right] g(x) \]

and

\[ |g(x) - F(x)| \leq \sum_{l_i = k_i}^{k_i+1} \cdots \sum_{l_n = k_n}^{k_n+1} \left[ \prod_{j=1}^{n} \mu_{A_j}^{x_j}(x_j) \right] |g(x) - g(\bar{x}^{(k_i, \ldots, k_n)})| \]

By using the mean value theorem for functions of several variables [31], we can derive the following inequality:

\[ |g(x) - g(\bar{x}^{(k_i, \ldots, k_n)})| \leq \left( \sup_{x \in U} \left[ \left[ \frac{\partial g}{\partial x_1} \cdots \frac{\partial g}{\partial x_n} \right] \right] \right) |x - \bar{x}^{(k_i, \ldots, k_n)}| \]

\[ = \left[ \left[ \frac{\partial g}{\partial x_1} \right]_{x=1} \cdots \left[ \frac{\partial g}{\partial x_n} \right]_{x=1} \right] \left[ \begin{array}{c} x_1 - \bar{x}^1 \\ \vdots \\ x_n - \bar{x}^n \end{array} \right] \]

\[ = \left[ \left[ \frac{\partial g}{\partial x_1} \right]_{x=1} \right] \left[ x_1 - \bar{x}^1 \right] + \cdots + \left[ \left[ \frac{\partial g}{\partial x_n} \right]_{x=1} \right] \left[ x_n - \bar{x}^n \right]. \]

Then, we can write (9) as

\[ |g(x) - F(x)| \leq \max_{l_j = k_j, k_j + 1} \left( \left[ \frac{\partial g}{\partial x_1} \right]_{x=1} \left[ x_1 - \bar{x}^1 \right] + \cdots + \left[ \frac{\partial g}{\partial x_n} \right]_{x=1} \left[ x_n - \bar{x}^n \right] \right). \]  

(10)

Because \( x_j \in [\bar{x}_j^{k_j}, \bar{x}_j^{k_j+1}] \), we know that \( \left| x_j - \bar{x}_j^{k_j} \right| \leq \left| \bar{x}_j^{k_j+1} - \bar{x}_j^{k_j} \right|. \) Hence, (10) implies

\[ |g(x) - F(x)| \leq \max_{1 \leq k_j, s \leq M_j - 1} \left( \left[ \frac{\partial g}{\partial x_1} \right]_{x=1} \left[ x_1^{k_j+1} - \bar{x}_1^{k_j} \right] + \cdots + \left[ \frac{\partial g}{\partial x_n} \right]_{x=1} \left[ x_n^{k_j+1} - \bar{x}_n^{k_j} \right] \right), \]

which in turn yields
\[
\left\| g(x) - F(x) \right\|_\infty \leq \left\| \frac{\partial g}{\partial x_1} \right\|_\infty \max_{1 \leq k \leq M_1-1} \left| x_1^{k+1} - x_1^k \right| + \cdots + \left\| \frac{\partial g}{\partial x_n} \right\|_\infty \max_{1 \leq k \leq M_n-1} \left| x_n^{k+1} - x_n^k \right|
\]

\[
= \left\| \frac{\partial g}{\partial x_1} \right\|_\infty h_1 + \cdots + \left\| \frac{\partial g}{\partial x_n} \right\|_\infty h_n
\]

where \( h_j = \max_{1 \leq k \leq M_j-1} \left| \bar{x}_j^{k+1} - \bar{x}_j^k \right| \) and \( 1 \leq j \leq n \).

Q.E.D.

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