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Feedback control of linear decentralized control systems: an algebraic approach

JIN S. LEE† and P. K. C. WANG‡

An algebraic approach to decentralized feedback control problems is considered in the framework of large-scale linear continuous systems. The problem considered is to determine conditions under which a controllable and observable linear system can be made controllable and observable from the input and output variables of a given control channel by static feedback applied to the other control channels. Then observer-controller or dynamic compensation scheme can be employed at this control channel in a standard way to make the whole closed-loop system stable.

1. Introduction

A fundamental result in modern linear control system theory is that the poles of a controllable linear system can be arbitrarily assigned by state variable feedback. This result was established by Wonham (1967).

Subsequently, it was shown that the poles of a closed-loop system consisting of a controllable and observable linear plant with an observer-based controller or with a dynamic compensator can be freely assigned (Luenberger 1964, Brasch and Pearson 1970).

All these results are based upon the assumption of centrality, that is, all the information available about the system and the decision-making based upon this information take place at a single location or channel. Thus the design leads to systems in which every sensor output affects every actuator input. This situation will be referred to as a system with centralized control.

However, in large-scale systems such as power systems, digital communication networks, economic systems, urban traffic networks and plasma control sytems, it is generally impossible to incorporate many feedback loops into the controller design and it is too costly even if it can be implemented.

This difficulty provides the motivation for the development of a decentralized control theory. The basic characteristic of decentralized control is that there are restrictions on the information transfer between certain groups of sensors and actuators. Thus, in a decentralized control systems, each channel generally does not have sufficient information to construct a feedback controller for the pole-placement or stabilization problem.

A natural generalization of the pole-placement or feedback stabilization problem to systems with decentralized feedback control has been considered by several authors (Aoki 1972, Wang and Davison 1973, Corfmat and Morse 1976 a, b). Among these

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works, Wang and Davison (1973) have considered the following problem. Under what condition can we find a decentralized local dynamic feedback control to stabilize the system?

A necessary and sufficient condition for the existence of a solution has been developed by Wang and Davison (1973), and it is stated in terms of 'fixed modes' which will be explained in § 3.

However, the construction of the local dynamic controllers suggested by Wang and Davison (1973) requires more integrators than necessary.

Motivated by this drawback, Corfmat and Morse have considered the following question. Under what condition can we find a decentralized local static feedback control to make the closed-loop system controllable and/or observable from a single channel? This is called the single-channel controllability and/or observability problem. If it is controllable and observable with respect to a single channel, then the standard observer-controller or dynamic compensator technique can be used to solve the pole placement problem of the decentralized control system.

A necessary and sufficient condition for the existence of a solution has been developed by Corfmat and Morse (1976 a, b) using a geometric approach.

From the practical point of view, this is more useful than the result given by Wang and Davison (1973), since static local feedback is used for all the channels except one. Moreover, this approach takes care of both the pole-placement as well as the stabilization problems.

The purpose of this work is to redevelop Corfmat and Morse's results from an algebraic approach. This approach is less technical and thus more transparent than the geometric approach given by Corfmat and Morse (1976 a, b).

The main reason for this approach is that it can easily be extended to systems with general delays in the state, and many of the results derived in this work are directly applicable with slight modifications to systems described by functional differential equations. These systems will be considered in future work (Lee 1984).

2. Problem formulation

Consider the linear time-invariant finite-dimensional system

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + \sum_{j=1}^{N} B_j u_j(t) \\ y_j(t) &= C_j x(t) \end{aligned}\right\} \qquad (1)$$

where $j = 1, 2, \ldots, N$ indexes the input and output variables of the various control channels; $u_j(t)$ and $y_j(t)$ are vectors of control and output variables associated with control channel j; $u_j \in \mathbb{R}^{m_j}$, $y_j \in \mathbb{R}^{p_j}$ and $x \in \mathbb{R}^n$ and A, B_j, C_j, $j = 1, 2, \ldots, N$ are constant $n \times n$, $n \times m_j$, $p_j \times n$ matrices representing linear maps.

In this paper, we consider a problem of determining conditions under which a system of the form (1) can be made controllable and observable from the input and output variables of a channel by static feedback applied by the other controllers. Then dynamic compensation can be employed at this channel in a standard way to place the poles of the system.

To avoid trivial situations, we assume that the system is completely controllable and observable at a fictitious centralized control and measurement station, i.e. $(A, [B_1 \;\; B_2 \;\; \ldots \;\; B_N])$ is completely controllable and $(A, [C_1 \;\; C_2 \;\; \ldots \;\; C_N])$ is completely observable. Moreover, to obtain an interesting problem, it is generally assumed that the system is not both completely controllable and observable from any one of the channels.

We begin by formulating the simpler problem of selecting an output feedback control $u_j(\cdot) = F_j y_j(\cdot)$ so that the resulting closed-loop system

$$\dot{x}(t) = \left(A + \sum_{\substack{j=1 \\ (j \neq i)}}^{N} B_j F_j C_j \right) x(t) + B_i u_i(t) \tag{2}$$

is completely controllable with $u_j(\cdot)$.

This problem is called the 'single-channel controllability problem'. When the full states of system (1) are unavailable from the output y_i, the observability condition of (2) is also necessary to design an observer to estimate all the states of (1).

Hence, the main problem in this section is to determine conditions in terms of the system matrices which characterizes (1) for the existence of an F_j, $j = 1, 2, \ldots, N$ such that (2) is controllable and observable with $u_i(\cdot)$. This problem will be referred to as the 'single-channel controllability and observability problem'. Closely related to this pole placement (or spectrum assignment) problem is the problem of determining when system (1) can be stabilized with decentralized control. Although this problem has been studied by Wang and Davison (1973) in a general setting, we restrict our attention to single-channel stabilizability and detectability problem of selecting $u_j(\cdot) = F_j y_j(\cdot)$ to make system (2) stabilizable and detectable with $u_i(\cdot)$.

This single-channel stabilizability and detectability approach is simple for implementation and can easily be extended to systems with general delays in the state. Results and proofs concerning this stabilizability problem are essentially the same as the pole placement problem and hence briefly mentioned in the sequel.

Notation

Here, $\mathbb{R}^n$ denotes the n-dimensional real linear vector space.

$$C^+ = \{\lambda \in C \mid \mathrm{Re}\,\lambda \geqslant 0\}, \quad C^- = \{\lambda \in C \mid \mathrm{Re}\,\lambda \leqslant 0\}$$

$\rho[A]$ is the rank of A and $\sigma(A) = \rho[\lambda I - A]$. For N-channel systems, we use the following notation: $\mathcal{N}$ denotes the set $\{1, 2, \ldots, N\}$ and $\mathcal{S}$ is a non-empty subset of $\mathcal{N}$ with elements $i_1, i_2, \ldots, i_s$ ordered so that $i_1 < i_2 < \ldots < i_s$. Then we define $B_{\mathcal{S}}$ and $C_{\mathcal{S}}$ so that

$$C_{\mathcal{S}} = \begin{bmatrix} C_{i_1} \\ C_{i_2} \\ \vdots \\ C_{i_s} \end{bmatrix} \quad \text{and} \quad B_{\mathcal{S}} = [B_{i_1} \quad B_{i_2} \quad \ldots \quad B_{i_s}]$$

Moreover, $\mathcal{P}(\mathcal{N})$ is a power set of $\mathcal{N}$, which is the set of all the subsets of $\mathcal{N}$,

$\mathcal{N}-\mathcal{S}=\{x|x\in\mathcal{N}$ and $x\notin\mathcal{S}\}$, $\mathcal{S}^+(i)$ denotes $\mathcal{S}\bigcup\{i\}$ with elements $i_1, i_2, \ldots, i, \ldots, i_s$ ordered so that $i_1<i_2<\ldots<i<\ldots<i_s$ and

$$C_{\mathcal{S}^+(i)}=\begin{bmatrix} C_{i_1} \\ C_{i_2} \\ \vdots \\ C_i \\ \vdots \\ C_{i_s} \end{bmatrix} \quad \text{and} \quad B_{\mathcal{S}^+(i)}=[B_{i_1} \quad B_{i_2} \quad \ldots \quad B_i \quad \ldots \quad B_{i_s}]$$

3. Preliminaries

In this section, we summarize some known results which will be used in the following sections in developing our results.

The following characterizations of controllability and observability are given by Hautus (1969).

Proposition 1

System (1) is completely controllable if and only if

$$\rho[\lambda I-A \quad B_1 \quad B_2 \quad \ldots \quad B_N]=n \quad \text{for all } \lambda\in\sigma(A)$$

Note that the condition $\lambda\in\sigma(A)$ above could be replaced by $\lambda\in \mathrm{C}$.

Proposition 2

System (1) is completely observable if and only if

$$\rho\begin{bmatrix} \lambda I-A \\ C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix}=n \quad \text{for all } \lambda\in\sigma(A)$$

An immediate consequence of the above two propositions is the following.

Proposition 3

System (1) is stabilizable (detectable) if and only if

$$\rho[\lambda I-A \quad B_1 \quad B_2 \quad \ldots \quad B_N]=n \quad \left(\rho\begin{bmatrix} \lambda I-A \\ C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix}=n\right) \quad \text{for all } \lambda\in C^+$$

The following proposition provides the solution to the problem of determining when system (1) can be stabilized with decentralized control.

Proposition 4 (Wang and Davison 1973)

For system (1), there exists a set of local feedback control laws such that the closed-loop system is asymptotically stable if and only if

$$\bigcap_{K_i \in \mathscr{K}_i} \sigma\left(A + \sum_{i=1}^{N} B_i K_i C_i\right) \subset C^- \quad \text{where } \mathscr{K}_i \subset \mathbb{R}^{m_i \times p_i}$$

$\bigcup_{K_i \in \mathscr{K}_i} \sigma\left(A + \sum_{i=1}^{N} B_i K_i C_i\right)$ is usually called the set of fixed modes.

The following proposition discovered by Anderson and Clements (1981) is extensively used for the development of our results and it is stated here without proof.

Proposition 5

Let $A_1, A_2, \ldots, A_m$ be m (real or complex) matrices of size $\alpha \times \beta_i$ and let $B_1, B_2, \ldots, B_m$ be m (real or complex) matrices of size $\alpha \times \gamma_i$.

Then, given $\varepsilon \geqslant 0$ and $\delta \geqslant 0$,

$$\rho[A_1 + B_1 K_1 \quad A_2 + B_2 K_2 \quad \ldots \quad A_m + B_m K_m] < \min\left\{\alpha - \delta, \sum_{i=1}^{m} \beta_i = \varepsilon\right\}$$

for all K_i of size $\gamma_i \times \beta_i$ if and only if there exists a non-empty subset $I = \{i_1, i_2, \ldots, i_j\}$ of $\{1, 2, \ldots, m\}$ for which

$$\rho[A_{i_1} \quad B_{i_1} \quad A_{i_2} \quad B_{i_2} \quad \ldots \quad A_{i_j} B_{i_j}] < \min\left\{\alpha - \delta - \sum_{i \in I} \beta_i, \sum_{i \in I} \beta_i - \varepsilon\right\}$$

Anderson and Clements (1981) have discussed further properties of fixed modes based on the above proposition and have discovered an interesting algebraic characterization of the fixed nodes, which is given below.

Proposition 6

For system (1), given $\delta > 0$,

$$\rho\left[\lambda I - A - \sum_{j=1}^{N} B_j F_j C_j\right] = n - \delta \quad \text{for all } F_j \in \mathbb{R}^{m_j \times p_j}$$

if and only if

$$\rho\begin{bmatrix} \lambda I - A & B_{\mathscr{S}} \\ C_{\mathscr{N} - \mathscr{S}} & 0 \end{bmatrix} < n - \delta \quad \text{for all } \mathscr{S} \in \mathscr{P}(\mathscr{N})$$

Next, we give a trivial extension of the result by Davison and Wang (1974). This result is originally concerned with the transmission zeros and is frequently used later in this work.

Proposition 7

Let A, B, and C be $n \times n$, $n \times m$, and $p \times n$ matrices, respectively, and

$$\max_{\lambda \in C} \rho \begin{bmatrix} \lambda I - A & B \\ C & 0 \end{bmatrix} = n + h, \qquad \text{where } h \leqslant \min \{m, p\}.$$

then the set

$$\bar{U} = \left\{ \lambda \in C \,\middle|\, \rho \begin{bmatrix} \lambda I - A & B \\ C & 0 \end{bmatrix} < n + h \right\}$$

is finite and the number of λ in $\bar{U}$ is less than or equal to $n - \max \{m, p\}$.

The next proposition, a result from matrix theory, is frequently used to prove the theorems in this work.

Proposition 8: *Sylvester's inequality*

If A is $m \times n$ and B *is* $n \times p$, then

$$\rho(A) + \rho(B) - n \leqslant \rho(AB) \leqslant \min \{\rho(A), \rho(B)\}$$

Finally, we introduce a definition of a robust subset for later references.

Definition 1 (Potter *et al.* 1979)

A subset of $\mathbb{R}^{m \times p}$ (respectively $C^{m \times p}$) is a *robust subset* (i.e. Zariski open set) of $\mathbb{R}^{m \times p}$ (respectively $C^{m \times p}$) if it is non-empty and if its complement is the set of solutions in $\mathbb{R}^{m \times p}$ (respectively $C^{m \times p}$) to a finite set of polynomial equations. Such sets are open and dense in $\mathbb{R}^{m \times p}$ (respectively $C^{m \times p}$), and each robust subset of $C^{m \times p}$ contains a largest subset which is a robust subset of $\mathbb{R}^{m \times p}$. The intersection of two robust subsets of $\mathbb{R}^{m \times p}$ (respectively $C^{m \times p}$) is also robust in $\mathbb{R}^{m \times p}$ (respectively $C^{m \times p}$). Any union of robust subsets of $\mathbb{R}^{m \times p}$ (respectively $C^{m \times p}$) is also robust in $\mathbb{R}^{m \times p}$ (respectively $C^{m \times p}$).

4. Single-channel controllability and observability of a two-channel system

The model to be used in this section is a two-channel linear time-invariant system described by

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 u_1(t) + B_2 U_2(t) \\ y_1(t) &= C_1 x(t), \quad y_2(t) = C_2 x(t) \end{aligned}\right\} \tag{3}$$

where $A \in \mathbb{R}^{n \times n}$, $B_2 \in \mathbb{R}^{n \times m_2}$, $C_1 \in \mathbb{R}^{p_1 \times n}$ and $C_2 \in \mathbb{R}^{p_2 \times n}$.

The problem to be considered first is to find conditions in terms of A, B_1, B_2, and C_2 for the existence of feedback matrix $F \in \mathbb{R}^{m_2 \times p_2}$ such that the system

$$\left.\begin{aligned} \dot{x}(t) &= (A + B_2 F C_2)x(t) + B_1 u(t) \\ y_1(t) &= C_1 x(t) \end{aligned}\right\} \tag{4}$$

is controllable from channel 1.

To avoid trivial cases, we assume that $B_1 \neq 0$, $B_2 \neq 0$, $C_1 \neq 0$, and $C_2 \neq 0$.

The answer to this problem is given in Theorem 1, the proof of which is based on a series of lemmas developed below.

Lemma 1

$C_2(\lambda I - A)^{-1}B_1 \neq 0$ over the field of rational functions in λ, if and only if there exists a $\lambda \in C$ such that

$$\rho\begin{bmatrix} \lambda I - A & B_1 \\ C_2 & 0 \end{bmatrix} > n$$

Proof

Necessity. Using the identity,

$$\begin{bmatrix} \lambda I - A & B_1 \\ C_2 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ C_2(\lambda I - A)^{-1} & I \end{bmatrix}\begin{bmatrix} \lambda I - A & B_1 \\ 0 & -C_2(\lambda I - A)^{-1}B_1 \end{bmatrix}, \quad \text{for } \lambda \notin \sigma(A)$$

We know that

$$\rho\begin{bmatrix} \lambda I - A & B_1 \\ C_2 & 0 \end{bmatrix} = \rho\begin{bmatrix} \lambda I - A & B_1 \\ 0 & -C_2(\lambda I - A)^{-1}B_1 \end{bmatrix}, \quad \text{for } \lambda \notin \sigma(A)$$

However, $C_2(\lambda I - A)^{-1}B_1 \neq 0$ over the field of rational functions in λ implies that there exists a $\lambda \in C - \sigma(A)$ such that $\rho[C_2(\lambda I - A)^{-1}B_1] > 0$.

For this λ,

$$\rho\begin{bmatrix} \lambda I - A & B_1 \\ C_2 & 0 \end{bmatrix} = \rho\begin{bmatrix} \lambda I - A & B_1 \\ 0 & -C_2(\lambda I - A)^{-1}B_1 \end{bmatrix} > n$$

Sufficiency. Assume $C_2(\lambda I - A)^{-1}B_1 = 0$ over the field of rational functions in λ and there exists a $\lambda \in C$ such that

$$\rho\begin{bmatrix} \lambda I - A & B_1 \\ C_2 & 0 \end{bmatrix} = n + h, \quad h \geqslant 1$$

Then, by Proposition 7,

$$\rho\begin{bmatrix} \lambda I - A & B_1 \\ C_2 & 0 \end{bmatrix} = n + h \tag{5}$$

for all but finitely many values of λ. However, for $\lambda \notin \sigma(A)$, we have the identity

$$\begin{bmatrix} \lambda I - A & B_1 \\ C_2 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ C_2(\lambda I - A)^{-1} & I \end{bmatrix}\begin{bmatrix} \lambda I - A & B_1 \\ 0 & -C_2(\lambda I - A)^{-1}B_1 \end{bmatrix}$$

Hence

$$\rho\begin{bmatrix} \lambda I - A & B_1 \\ C_2 & 0 \end{bmatrix} = \rho\begin{bmatrix} \lambda I - A & B_1 \\ 0 & -C_2(\lambda I - A)^{-1}B_1 \end{bmatrix} = n \tag{6}$$

for $\lambda \notin \sigma(A)$. (Since $C_2(\lambda I - A)^{-1}B_1 = 0$.) So (5) and (6) imply a contradiction. Hence, $C_2(\lambda I - A)^{-1}B_1 = 0$ implies

$$\rho\begin{bmatrix} \lambda I - A & B_1 \\ C_2 & 0 \end{bmatrix} \leqslant n \quad \text{for all } \lambda \in C$$

Therefore, sufficiency is proved. □

Lemma 2

Let Σ be any finite set of complex numbers. If

$$\rho\begin{bmatrix} \lambda I - A & B_1 \\ C_2 & 0 \end{bmatrix} \geqslant n \quad \text{for all } \lambda \in \sigma(A)$$

and

$$\rho[\lambda I - A \quad B_1 \quad B_2] = n \quad \text{for all } \lambda \in \sigma(A)$$

then the set $\mathscr{F} = \{F \in \mathbb{R}^{m_2 \times p_2} | \rho[\lambda I - A - B_2 F C_2 \quad B_1] = n \text{ for all } \lambda \in \Sigma\}$ is robust in $\mathbb{R}^{m_2 \times p_2}$.

Proof

For any fixed $\lambda \in \Sigma$, we show that there exists a matrix F_λ such that $\rho[\lambda I - A - B_2 F_\lambda C_2 \quad B_1] = n$.

Assume that $\rho[\lambda I - A - B_2 F_\lambda C_2 \quad B_1] < n$ for all $F_\lambda \in \mathbb{R}^{m_2 \times p_2}$ and let $\pi = \max\{m_2, p_2\}$. By adding columns of zeros to B_2 when $p_2 > m_2$ and rows of zeros to C_2 when $p_2 < m_2$, we can form matrices $\bar{B}_2$ and $\bar{C}_2$ with π columns and rows, respectively.

Then

$$\rho[\lambda I - A - B_2 F C_2 \quad B_1] < n \quad \text{for all } F \in \mathbb{R}^{m_2 \times p_2}$$

implies $\rho[\lambda I - A - \bar{B}_2 \bar{F} \bar{C}_2 \quad B_1] < n$ for all non-singular $\bar{F} \in \mathbb{R}^{\pi \times \pi}$ (Anderson and Clements 1981). From this, it follows that

$$\rho\begin{bmatrix} \lambda I - A - \bar{B}_2 \bar{F} \bar{C}_2 & 0 & B_1 \\ \bar{C}_2 & (\bar{F})^{-1} & 0 \end{bmatrix} < n + \pi$$

for all non-singular $\bar{F} \in \mathbb{R}^{\pi \times \pi}$. However, since

$$\begin{bmatrix} I & -\bar{B}_2 \bar{F} \\ 0 & I \end{bmatrix}\begin{bmatrix} \lambda I - A & \bar{B}_2 & B_1 \\ \bar{C}_2 & (\bar{F})^{-1} & 0 \end{bmatrix} = \begin{bmatrix} \lambda I - A - \bar{B}_2 \bar{F} \bar{C}_2 & 0 & \bar{B}_1 \\ \bar{C}_2 & (\bar{F})^{-1} & 0 \end{bmatrix}$$

holds, we have that

$$\rho\begin{bmatrix} \lambda I - A & \bar{B}_2 & B_1 \\ \bar{C}_2 & (\bar{F})^{-1} & 0 \end{bmatrix} < n + \pi$$

for all non-singular $\bar{F} \in \mathbb{R}^{\pi \times \pi}$. This implies that the above inequality holds for all

$\bar{F} \in \mathbb{R}^{\pi \times \pi}$. If we set

$$\tilde{A}_1 = \begin{bmatrix} \lambda I - A \\ \bar{C}_2 \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix} \bar{B}_2 \\ 0 \end{bmatrix}, \quad \tilde{B}_2 = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

$$\tilde{A}_3 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad \tilde{B}_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \tilde{K}_2 = (\bar{F})^{-1}$$

then the above inequality is represented as

$$\rho[\tilde{A}_1 + \tilde{B}_1 \tilde{K}_1 \quad \tilde{A}_2 + \tilde{B}_2 \tilde{K}_2 \quad \tilde{A}_3 + \tilde{B}_3 \tilde{K}_3] < n + \pi$$

for all $\tilde{K}_1, \tilde{K}_2, \tilde{K}_3$.

By using Proposition 5, it follows that there exists a non-empty subset I of $\{1, 2, 3\}$ for which

$$\rho[\tilde{A}_1 \quad \tilde{B}_1 \quad \tilde{A}_2 \quad \tilde{B}_2 \quad \tilde{A}_3 \quad \tilde{B}_3] = \rho \begin{bmatrix} \lambda I - A & \bar{B}_2 & 0 & B_1 \\ \bar{C}_2 & 0 & I & 0 \end{bmatrix}$$

$$< \min \left\{ n + \pi - \sum_{i \notin I} \beta_i, \sum_{i \in I} \beta_i \right\}$$

where $\beta_1 = n$, $\beta_2 = \pi$. $\beta_3 = m_1$. Considering all subsets I of $\{1, 2, 3\}$, the above inequality leads to

$$\rho \begin{bmatrix} \lambda I - A & B_1 \\ C_2 & 0 \end{bmatrix} < n \quad \text{or} \quad \rho[\lambda I - A \quad B_1 \quad B_2] < n$$

which is contrary to our assumptions. Hence, there exists a matrix $F_\lambda \in \mathbb{R}^{m_2 \times p_2}$ such that $\rho[\lambda I - A - B_2 F_\lambda C_2 \quad B_1] = n$.

The preceding statements imply that for each fixed λ, the vector $z_\lambda(F)$ of all nth-order minors of $[\lambda I - A - B_2 F C_2 \quad B_1]$ must be non-zero at $F = F_\lambda$. Since, in addition, the elements of $z_\lambda(F)$ are polynomial functions of the elements of F, $z_\lambda(F)$ is not identically zero. This proves that $\mathscr{F}_\lambda = \{F \in C^{m_2 \times p_2} | \rho[\lambda I - A - B_2 F C_2 \quad B_1] = n\}$ is robust in $C^{m_2 \times p_2}$ and so is $\tilde{\mathscr{F}} = \bigcap_{\lambda \in \Sigma} \mathscr{F}_\lambda$.

Hence, if we define $\mathscr{F}$ to be the largest subset of $\tilde{\mathscr{F}}$ which is a robust subset of $\mathbb{R}^{m_2 \times p_2}$, then $\mathscr{F}$ has the required property. □

Lemma 3

Let

$$\max_{\lambda \in C} \rho \begin{bmatrix} \lambda I - A & B_1 \\ C_2 & 0 \end{bmatrix} = n + h$$

where $h \geqslant 1$, then the set

$$\mathscr{K} = \left\{ K \in \mathbb{R}^{h \times p_2} \middle| \max_{\lambda \in C} \rho \begin{bmatrix} \lambda I - A & B_1 \\ K C_2 & 0 \end{bmatrix} = n + h \right\}$$

is robust in $\mathbb{R}^{h \times p_2}$.

Proof

Let

$$\bar{U} = \left\{ \lambda \in C \,\middle|\, \rho \begin{bmatrix} \lambda I - A & B_1 \\ C_2 & 0 \end{bmatrix} = n + h \ (h \geqslant 1) \right\}$$

and fix $\lambda \in \bar{U}$. Then we can find a $K_\lambda \in \mathbb{R}^{h \times p_2}$ such that

$$\rho \begin{bmatrix} \lambda I - A & B_1 \\ K_\lambda C_2 & 0 \end{bmatrix} = n + h$$

In fact, if

$$\rho \begin{bmatrix} \lambda I - A & B_1 \\ K_\lambda C_2 & 0 \end{bmatrix} = \rho \begin{bmatrix} \lambda I - A^{\mathrm{T}} & C_2^{\mathrm{T}} K_\lambda^{\mathrm{T}} \\ B_1^{\mathrm{T}} & 0 \end{bmatrix} < n + h \quad \text{for all } K_\lambda \in \mathbb{R}^{h \times p_2}$$

then, by substituting

$$\tilde{A}_1 = \begin{bmatrix} \lambda I - A^{\mathrm{T}} \\ B_1^{\mathrm{T}} \end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \tilde{B}_2 = \begin{bmatrix} C_2^{\mathrm{T}} \\ 0 \end{bmatrix}$$

into Proposition 5, we have

$$\rho \begin{bmatrix} \lambda I - A^{\mathrm{T}} & C_2^{\mathrm{T}} \\ B_1^{\mathrm{T}} & 0 \end{bmatrix} = \rho \begin{bmatrix} \lambda I - A & B_1 \\ C_2 & 0 \end{bmatrix} < n + h$$

which is a contradiction.

Since we have proved the existence of a K_λ that satisfies

$$\rho \begin{bmatrix} \lambda I - A & B_1 \\ K_\lambda C_2 & 0 \end{bmatrix} = n + h$$

the robustness of

$$\bar{\mathscr{K}}_\lambda = \left\{ K_\lambda \in \mathscr{R}^{k \times p_2} \,\middle|\, \rho \begin{bmatrix} \lambda I - A & B_1 \\ K_\lambda C_2 & 0 \end{bmatrix} = n + h \right\}$$

can be established using the same argument as in Lemma 2. Hence,

$$\bar{\mathscr{K}} = \left\{ K \in \mathbb{R}^{h \times p_2} \,\middle|\, \max_{\lambda \in C} \rho \begin{bmatrix} \lambda I - A & B_1 \\ K C_2 & 0 \end{bmatrix} = n + h \right\} = \bigcup_{\lambda \in \bar{U}} \bar{\mathscr{K}}_\lambda$$

is also robust in $\mathbb{R}^{h \times p_2}$ from the fact that any union of robust subsets of $\mathbb{R}^{h \times p_2}$ is also robust in $\mathbb{R}^{h \times p_2}$. □

Lemma 4

Let

$$\rho \begin{bmatrix} \lambda I - A & B_1 \\ C_2 & 0 \end{bmatrix} \geqslant n \quad \text{for all } \lambda \in C$$

Then the set

$$\tilde{\mathscr{K}} = \left\{ K \in \mathbb{R}^{h \times p_2} \middle| \rho \begin{bmatrix} \lambda I - A & B_1 \\ KC_2 & 0 \end{bmatrix} \geqslant n \text{ for all } \lambda \in C \right\}$$

is robust in $\mathbb{R}^{h \times p_2}$.

Proof

When $\lambda \in C - \sigma(A)$,

$$\rho \begin{bmatrix} \lambda I - A & B_1 \\ KC_2 & 0 \end{bmatrix} \geqslant n \quad \text{for all } K \in \mathbb{R}^{h \times p_2}$$

When $\lambda \in \sigma(A)$, we fix $\lambda \in \sigma(A)$. Following the similar argument as in the proof Lemma 2, it follows that the set

$$\mathscr{K}_\lambda = \left\{ K \in \mathbb{R}^{h \times p_2} \middle| \rho \begin{bmatrix} \lambda I - A & B_1 \\ KC_2 & 0 \end{bmatrix} \geqslant n \right\}$$

is robust in $\mathbb{R}^{h \times p_2}$ and so is $\tilde{\mathscr{K}} = \bigcap_{\lambda \in \sigma(A)} \mathscr{K}_\lambda$. □

Theorem 1

For the multivariable finite-dimensional linear time-invariant system (3), there exists a feedback matrix $F \in \mathbb{R}^{m_2 \times p_2}$ such that

$$\rho[\lambda I - A - B_2 F C_2 \quad B_1] = n \quad \text{for all } \lambda \in C \tag{7}$$

if and only if $C_2(\lambda I - A)^{-1} B_1 \neq 0$ over the field of rational functions in λ,

$$\rho[\lambda I - AB_1 \quad B_2] = n \quad \text{for all } \lambda \in \sigma(A) \tag{8}$$

and

$$\rho \begin{bmatrix} \lambda I - A & B_1 \\ C_2 & 0 \end{bmatrix} \geqslant n \quad \text{for all } \lambda \in \sigma(A) \tag{9}$$

Moreover, the set $\mathscr{F} = \{F \in \mathbb{R}^{m_2 \times p_2} | \rho[\lambda I - A - B_2 F C_2 \quad B_1] = n, \text{ for all } \lambda \in C\}$ is robust.

Proof

Necessity. First we prove that (7) implies $C_2(\lambda I - A)^{-1} B_1 \neq 0$. Assume that $C_2(\lambda I - A)^{-1} B_1 = 0$, which implies $C_2 A^i B_1 = 0$ for all $i \geqslant 0$. Thus

$$C_2[B_1 \quad AB_1 \quad \dots \quad A^{n-1} B_1] = 0 \tag{10}$$

However,

$$\rho[\lambda I - A - B_2 F C_2 \quad B_1] = n \quad \text{for all } \lambda \in C$$

implies that

$$n = \rho[B_1 \quad (A + B_2 F C_2) B_1 \quad \dots \quad (A + B_2 F C_2)^{n-1} B_1] = \rho[B_1 \quad AB_1 \quad \dots \quad A^{n-1} B_1] \tag{11}$$

From (10) and (11), we have $C_2 = 0$ which is contrary to our assumption. Hence, $C_2(\lambda I - A)^{-1}B_1 \neq 0$.

Next, from the matrix identities

$$[\lambda I - A - B_2FC_2 \quad B_1] = [\lambda I - AB_1 \quad B_2]\begin{bmatrix} I & 0 \\ 0 & I \\ -FC_2 & 0 \end{bmatrix}$$

$$[\lambda I - A - B_2FC_2 \quad B_1] = [I - B_2F]\begin{bmatrix} \lambda I - A & B_1 \\ C_2 & 0 \end{bmatrix}$$

and Sylvester's inequality (Proposition 8), it follows that (8) and (9) hold.

Sufficiency. We see that $C_2(\lambda I - A)^{-1}B_1 \neq 0$ implies that

$$\max_{\lambda \in C} \rho \begin{bmatrix} \lambda I - A & B_1 \\ C_2 & 0 \end{bmatrix} = n + h \quad (h \geqslant 1)$$

by Lemma 1.

Then, from this condition and Lemma 3, the set

$$\bar{\mathscr{K}} = \left\{ K \in \mathbb{R}^{h \times p_2} \middle| \max_{\lambda \in C} \rho \begin{bmatrix} \lambda I - A & B_1 \\ KC_2 & 0 \end{bmatrix} = n + h \right\} \tag{12}$$

is robust in $\mathbb{R}^{h \times p_2}$. From (9) and Lemma 4, the set

$$\tilde{\mathscr{K}} = \left\{ K \in \mathbb{R}^{h \times p_2} \middle| \rho \begin{bmatrix} \lambda I - A & B_1 \\ KC_2 & 0 \end{bmatrix} \geqslant n \quad \text{for all } \lambda \in C \right\} \tag{13}$$

is robust in $\mathbb{R}^{h \times p_2}$ and so is $\mathscr{K} = \bar{\mathscr{K}} \bigcap \tilde{\mathscr{K}}$.

Now, let us take any $K \in \mathscr{K} \subset \mathbb{R}^{h \times p_2}$. From (12) and Proposition 7, we have that

$$\rho \begin{bmatrix} \lambda I - A & B_1 \\ KC_2 & 0 \end{bmatrix} = n + h$$

for all $\lambda \in C - \Sigma$ where Σ is finite. Now, for $\lambda \in C - \Sigma$, it follows from Sylvester's inequality that

$$\rho[\lambda I - A - B_2GKC_2B_1] \geqslant \rho[I - B_2G] + \rho \begin{bmatrix} \lambda I - A & B_1 \\ KC_2 & 0 \end{bmatrix} - n - h$$

$$= n + n + h - n - h = n \quad \text{for all } G \in \mathbb{R}^{m_2 \times h}$$

On the other hand, from (13) and Lemma 2, the set

$$\mathscr{G} = \{G \in \mathbb{R}^{m_2 \times h} | \rho[\lambda I - A - B_2GKC_2 \quad B_1] = n, \text{ for all } \lambda \in \Sigma\}$$

is robust. Hence, if we take any G from this set $\mathscr{G}$, then $\rho[\lambda I - A - B_2GKC_2B_1] = n$, for all $\lambda \in C$.

Therefore this GK is the feedback controller we are looking for and the existence is proved.

The preceding implies that $\rho[B_1 \quad (A+B_2FC_2)B_1 \quad \dots \quad (A+B_2FC_2)^{n-1}B_1]=n$ at $F=GK$. Hence, the vector $z(F)$ of all nth-order minors of $[B_1 \quad (A+B_2FC_2)B_1 \quad \dots \quad (A+B_2FC_2)^{n-1}B_1]$ must be non-zero at $F=GK$. Since, in addition, the elements of $z(F)$ are polynomial functions of the elements of F, $z(F)$ is not identically zero. This proves that

$$\begin{aligned}\mathscr{F} &= \{F \in \mathbb{R}^{m_2 p_2} \mid \rho[B_1 \quad (A+B_2FC_2)B_1 \quad \dots \quad (A+B_2FC_2)^{n-1}B_1]=n\} \\ &= \{F \in \mathbb{R}^{m_2\times p_2} \mid \rho[\lambda I - A - B_2FC_2 \quad B_1]=n\}\end{aligned}$$

is robust. □

As a corollary to Theorem 1, we can derive conditions for the existence of a feedback matrix $F_2 \in \mathbb{R}^{m_2\times p_2}$ such that system (7) is stabilizable on channel 1.

Corollary 1

Assume $C_2(\lambda I - A)^{-1}B_1 \neq 0$ for system (3). There exists a feedback matrix $F \in \mathbb{R}^{m_2\times p_2}$ such that

$$\rho[\lambda I - A - B_2FC_2 \quad B_1]=n \quad \text{for all } \lambda \in C^+$$

if and only if

$$\rho[\lambda I - A \quad B_1 \quad B_2]=n \quad \text{for all } \lambda \in C^+ \bigcap \sigma(A)$$

and

$$\rho\begin{bmatrix}\lambda I - A & B_1 \\ C_2 & 0\end{bmatrix} \geqslant n \quad \text{for all } \lambda \in C^+ \bigcap \sigma(A)$$

Remark 1

The same proof as that of Theorem 1 applies to this corollary with $\lambda \in C$ and $\lambda \in \sigma(A)$ in Theorem 1 replaced by $\lambda \in C^+$ and $\lambda \in C^+ \bigcap \sigma(A)$, respectively.

The next problem to be considered based on model (3) is to derive conditions for the existence of a feedback matrix $F_2 \in \mathbb{R}^{m_2\times p_2}$ such that system (7) is completely controllable and observable with respect to channel 1.

Theorem 2

For system (3), there exists a feedback matrix $F \in \mathbb{R}^{m_2\times p_2}$ such that

$$\rho[\lambda I - A - B_2FC_2 \quad B_1]=n \quad \text{for all } \lambda \in C$$

and

$$\rho\begin{bmatrix}\lambda I - A - B_2FC_2 \\ C_1\end{bmatrix} = n \quad \text{for all } \lambda \in C$$

if and only if

$$C_2(\lambda I - A)^{-1}B_1 \neq 0, \quad C_1(\lambda I - A)^{-1}B_2 \neq 0$$

$$\rho[\lambda I - AB_1 \quad B_2] = n \quad \text{for all } \lambda \in \sigma(A)$$

$$\rho\begin{bmatrix} \lambda I - A \\ C_1 \\ C_2 \end{bmatrix} = n \quad \text{for all } \lambda \in \sigma(A)$$

$$\rho\begin{bmatrix} \lambda I - A & B_1 \\ C_2 & 0 \end{bmatrix} \geqslant n \quad \text{for all } \lambda \in \sigma(A)$$

and

$$\rho\begin{bmatrix} \lambda I - A & B_2 \\ C_1 & 0 \end{bmatrix} \geqslant n \quad \text{for all } \lambda \in \sigma(\mathrm{A})$$

Proof

From Theorem 1 and its duality, it follows that both

$$\mathscr{F}_1 = \{F \in \mathbb{R}^{m_2 \times p_2} | \rho[\lambda I - A - B_2FC_2 \quad B_1] = n \quad \text{for all } \lambda \in C\}$$

and

$$\mathscr{F}_2 = \left\{F \in \mathbb{R}^{m_2 \times p_2} \middle| \rho\begin{bmatrix} \lambda I - A - B_2FC_2 \\ B_1 \end{bmatrix} = n \quad \text{for all } \lambda \in C\right\}$$

are robust and so is $\mathscr{F} = \mathscr{F}_1 \bigcap \mathscr{F}_2$.

If we take any $F \in \mathscr{F}$, this F makes system (4) completely controllable and observable. □

Corollary 2

Assume $C_2(\lambda I - A)^{-1}B_1 \neq 0$ and $C_1(\lambda I - A)^{-1}B_2 \neq 0$ for system (3). There exists a feedback matrix $F \in \mathbb{R}^{m_2 \times p_2}$ such that system (4) is stabilizable and detectable from channel 1 if and only if system (3) is stabilizable and detectable and

$$\rho\begin{bmatrix} \lambda I - A & B_1 \\ C_2 & 0 \end{bmatrix} \geqslant n \quad \text{for } \lambda \in C^+ \bigcap \sigma(A)$$

and

$$\rho\begin{bmatrix} \lambda I - A & B_2 \\ C_1 & 0 \end{bmatrix} \geqslant n \quad \text{for } \lambda \in C^+ \bigcap \sigma(A)$$

5. Single-channel controllability and observability of an N-channel system

We are now in a position to extend the results developed in § 4 to the N-channel case. The notation defined in § 2 is extensively used in this section. The model to be used is given in (1) and the problem to be considered is to determine conditions in terms of the system parameters to select a feedback matrix $F_j, j = 1, 2, \ldots, N$ so that system (2) is completely controllable with respect to channel j. Later, it is generalized to the problem of selecting a feedback matrix $F_j, j = 1, 2, \ldots, N$ to make system (2) completely controllable and observable with respect to channel j. The single-channel stabilizability problem and stabilizability and observability problem can be solved in a similar way and they are mentioned briefly in the Remark 1.

To avoid trivial cases, we again assume that $C_j \neq 0, B_j \neq 0$ for all $j = 1., 2, \ldots, N$.

Theorem 3

For the multivariable finite-dimensional linear time-invariant system (1), there exist feedback matrices $F_j \in \mathbb{R}^{m_j \times p_j}, j = 1, 2, \ldots, N$ such that

$$\rho\left[\lambda I - A - \sum_{j=1}^{N} B_j F_j C_j \quad B_i\right] = n \quad \text{for all } \lambda \in C \tag{14}$$

if and only if for all $\mathscr{S} \in P(N)$,

$$C_{\mathscr{N}-\mathscr{S}^*(i)}(\lambda I - A)^{-1} B_{\mathscr{S}^*(i)} \neq 0$$

and

$$\rho\begin{bmatrix} \lambda I - A & B_{\mathscr{S}^*(i)} \\ C_{\mathscr{N}-\mathscr{S}^*(i)} & 0 \end{bmatrix} \geqslant n \quad \text{for all } \lambda \in \sigma(A)$$

Proof

Necessity. This part is an extension of the two-channel case as in the proof of Theorem 1. First, assume $C_{\mathscr{N}-\mathscr{S}^*(i)}(\lambda I - A)^{-1} B_{\mathscr{S}^*(i)} = 0$ which implies $C_{\mathscr{N}-\mathscr{S}^*(i)} A^l B_{\mathscr{S}^*(i)} = 0$ for $l \geqslant 0$. Thus

$$C_{\mathscr{N}-\mathscr{S}^*(i)}[B_{\mathscr{S}^*(i)} \quad AB_{\mathscr{S}^*(i)} \quad \ldots \quad A^{n-1} B_{\mathscr{S}^*(i)}] = 0 \tag{15}$$

However, (14) implies

$$\rho\left[\lambda I - A - \sum_{j=1}^{N} B_j F_j C_j \quad B_{\mathscr{S}^*(i)}\right] = n$$

which is equivalent to saying that

$$\left.\begin{aligned} n &= \rho\left[B_{\mathscr{S}^*(i)}\left(A + \sum_{j=1}^{N} B_j F_j C_j\right) B_{\mathscr{S}^*(i)} \quad \ldots \quad \left(A + \sum_{j=1}^{N} B_j F_j C_j\right)^{n-1} B_{\mathscr{S}^*(i)}\right] \\ &= \rho[B_{\mathscr{S}^*(i)} \quad AB_{\mathscr{S}^*(i)} \quad \ldots \quad A^{n-1} B_{\mathscr{S}^*(i)}], \\ &\quad \text{since } C_{\mathscr{N}-\mathscr{S}^*(i)} A^l B_{\mathscr{S}^*(i)} = 0 \quad \text{for } l \geqslant 0 \end{aligned}\right\} \tag{16}$$

From (15) and (16), we have $C_{\mathscr{N}-\mathscr{S}^*(i)} = 0$, contrary to our assumption. Hence it follows that

$$C_{\mathscr{N}-\mathscr{S}^*(i)}(\lambda I - A)^{-1} B_{\mathscr{S}^*(i)} \neq 0 \quad \text{for all } \mathscr{S} \in \mathscr{P}(\mathscr{N})$$

Next, if we define $F_{\mathscr{S}^*(i)} = \text{diag } [F_{i_1}, F_{i_2}, \ldots, F_i, \ldots, F_{i_s}]$ and $F_{\mathscr{N}-\mathscr{S}^*(i)} =$

diag $[F_{i_{s+1}}, \ldots, F_{i_N}]$, then we have the following matrix identity:

$$\left[\lambda I - A - \sum_{j=1}^{N} B_j F_j C_j \quad B_i\right] = \left[\lambda I - A - \sum_{\substack{j=1 \\ (j \neq i)}}^{N} B_j F_j C_j \quad B_i\right]$$

$$= [I - B_{\mathscr{S}^+(i)} F_{\mathscr{S}^+(i)}] \begin{bmatrix} \lambda I - A & B_{\mathscr{S}^+(i)} \\ C_{\mathscr{N} - \mathscr{S}^+(i)} & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} I & 0 \\ 0 & I \\ -F_{\mathscr{N}-\mathscr{S}^+(i)} C_{\mathscr{N}-\mathscr{S}^+(i)} & 0 \end{bmatrix} \quad \text{for all } \mathscr{S} \in \mathscr{P}(\mathscr{N})$$

From the above identity and Sylvester's inequality, it follows that

$$\rho \begin{bmatrix} \lambda I - A & B_{\mathscr{S}^+(i)} \\ C_{\mathscr{N} - \mathscr{S}^+(i)} & 0 \end{bmatrix} \geqslant n \quad \text{for all } \lambda \in \sigma(A) \quad \text{for all } \mathscr{S} \in \mathscr{P}(\mathscr{N})$$

Sufficiency. Mathematical induction is used for this sufficiency proof.

(*a*) When $N = 2$, the proof is given in Theorem 1.

(*b*) when the conclusion of the theorem is true for $N = k$, it will be shown that the same conclusion, also holds for $N = k + 1$.

Let $\Delta(k) = \lambda I - A$. Let $\mathscr{K} = \{1, 2, \ldots, k\}$ and $\mathscr{K}^* = \{1, 2, \ldots, k+1\}$. For all $\mathscr{S} \in P(\mathscr{K}^*)$, aassume that

$$C_{\mathscr{K}^* - \mathscr{S}^+(i)} \Delta(\lambda)^{-1} B_{\mathscr{S}^+(i)} \neq 0 \tag{17}$$

and

$$\rho \begin{bmatrix} \Delta(\lambda) & B_{\mathscr{S}^+(i)} \\ C_{\mathscr{K}^* - \mathscr{S}^+(i)} & 0 \end{bmatrix} \geqslant n \cdot \text{ for all } \lambda \in C^+ \tag{18}$$

Depending on whether $k + 1 \in \mathscr{K}^* - \mathscr{S}^+(i)$, or $k + 1 \in \mathscr{S}^+(i)$, (17) is divided into

$$C_{\mathscr{K} - \mathscr{S}^+(i)} \Delta(\lambda)^{-1} [B_{\mathscr{S}^+(i)} | B_{k+1}] \neq 0 \quad \text{and} \quad \begin{bmatrix} C_{\mathscr{K} - \mathscr{S}^+(i)} \\ C_{k+1} \end{bmatrix} \Delta(\lambda)^{-1} B_{\mathscr{S}^+(i)} \neq 0 \tag{19}$$

for all $\mathscr{S} \in P(\mathscr{K})$. In the same way, (18) is divided into

$$\rho \begin{bmatrix} \Delta(\lambda) & B_{\mathscr{S}^+(i)} & B_{k+1} \\ C_{\mathscr{K} - \mathscr{S}^+(i)} & 0 & 0 \end{bmatrix} \geqslant n \quad \text{and} \quad \rho \begin{bmatrix} \Delta(\lambda) & B_{\mathscr{S}^+(i)} \\ C_{\mathscr{K} - \mathscr{S}^+(i)} & 0 \\ C_{k+1} & 0 \end{bmatrix} \geqslant n \tag{20}$$

for all $\lambda \in C^+$ and for all $\mathscr{S} \in \mathscr{P}(\mathscr{K})$.

The second condition of (19), implies that

$$\min_{\mathscr{S}\in\mathscr{P}(\mathscr{K})} \max_{\lambda\in C^+} \rho\begin{bmatrix} \Delta(\lambda) & B_{\mathscr{S}^*(i)} \\ C_{\mathscr{K}-\mathscr{S}^*(i)} & 0 \\ C_{k+1} & 0 \end{bmatrix} = n+h, \quad h\geqslant 1$$

Hence, following the similar argument as in the proof of Lemmas 3 and 4, the conditions of (19) and (20) imply that for all $\mathscr{S}\in\mathscr{P}(\mathscr{K})$,

$$\mathscr{K}_{\mathscr{S}^*(i)} = \left\{ K\in\mathbb{R}^{h\times p_{k+1}} \;\middle|\; \rho\begin{bmatrix} \Delta(\lambda) & B_{\mathscr{S}^*(i)} \\ C_{\mathscr{K}-\mathscr{S}^*(i)} & 0 \\ KC_{k+i} & 0 \end{bmatrix} \right.$$

$$\left. \geqslant n, \text{ for all } \lambda\in C^+ \text{ and } \max_{\lambda\in C^+} \rho\begin{bmatrix} \Delta(\lambda) & B_{\mathscr{S}^*(i)} \\ C_{\mathscr{K}-\mathscr{S}^*(i)} & 0 \\ C_{k+1} & 0 \end{bmatrix} \geqslant n+h \right\}$$

is robust in $\mathbb{R}^{h\times p_{k+1}}$ and so is $\mathscr{K} = \bigcap_{\mathscr{S}\in\mathscr{P}(\mathscr{K})}\mathscr{K}_{\mathscr{S}^*(i)}$. If we take any $K\in\mathscr{K}\subset\mathbb{R}^{h\times p_{k+1}}$, it satisfies the condition that for all $\mathscr{S}\in\mathscr{P}(\mathscr{K})$,

$$\rho\begin{bmatrix} \Delta(\lambda) & B_{\mathscr{S}^*(i)} \\ C_{\mathscr{K}-\mathscr{S}^*(i)} & 0 \\ KC_{k+1} & 0 \end{bmatrix} \geqslant n+h \quad \text{for all } \lambda\in C^+-\Sigma \tag{21}$$

where Σ is finite, and

$$\rho\begin{bmatrix} \Delta(\lambda) & B_{\mathscr{S}^*(i)} \\ C_{\mathscr{K}-\mathscr{S}^*(i)} & 0 \\ KC_{k+1} & 0 \end{bmatrix} \geqslant n \quad \text{for all } \lambda\in C^+ \tag{22}$$

When $\lambda\in C^+-\Sigma$, it follows from Sylvester's inequality that

$$\rho\begin{bmatrix} \Delta(\lambda)-B_{k+1}GKC_{k+1} & B_{\mathscr{S}^*(i)} \\ C_{\mathscr{K}-\mathscr{S}^*(i)} & 0 \end{bmatrix}$$

$$\geqslant \rho\begin{bmatrix} I & 0 & -B_{k+1}G \\ 0 & I & 0 \end{bmatrix} + \rho\begin{bmatrix} \Delta(\lambda) & B_{\mathscr{S}^*(i)} \\ C_{\mathscr{K}-\mathscr{S}^*(i)} & 0 \\ KC_{k+1} & 0 \end{bmatrix} - n - w - h$$

$$\geqslant (n-w)+(n+h)-n-w-h$$

$$= n \text{ for all } \mathscr{S}\in\mathscr{P}(\mathscr{K}) \quad \text{and } \mathscr{G}\in\mathbb{R}^{m_{k+1}\times h}$$

Here w is the number of rows of $C_{\mathscr{K}-\mathscr{S}^*(i)}$.

When $\lambda\in\Sigma$, following the same reasoning as that in the proof of Lemma 2, the first conditions of (16) and (18) imply that there exists a $G \in \mathbb{R}^{m_{k+1} \times h}$ such that

$$\rho\begin{bmatrix} \Delta(\lambda) - \mathrm{B}_{k+1}\mathrm{G}\mathrm{K}\mathrm{C}_{k+1} & B_{\mathscr{S}^+(i)} \\ C_{\mathscr{K}-\mathscr{S}^+(i)} & 0 \end{bmatrix} \geqslant n \quad \text{for all } \lambda \in \Sigma, \quad \text{for all } \mathscr{S} \in P(\mathscr{K})$$

From the above and the robustness argument as in the proof of Lemma 3, it is evident that the set

$$\tilde{\mathscr{G}} = \left\{ G \in \mathbb{R}^{m_{k+1} \times h} \,\middle|\, \rho\begin{bmatrix} \Delta(\lambda) - B_{k+1}GKC_{k+1} & B_{\mathscr{S}^+(i)} \\ C_{\mathscr{K}-\mathscr{S}^+(i)} & 0 \end{bmatrix} \geqslant n, \quad \text{for all } \lambda \in C^+ \text{ and for all } \mathscr{S} \in P(\mathscr{K}) \right\}$$

is robust in $\mathbb{R}^{m_{k+1} \times h}$.

On the other hand, the first condition of (19) implies that for all $\mathscr{S} \in \mathscr{P}(\mathscr{K})$,

$$\rho\begin{bmatrix} \Delta(\lambda_0) & B_{\mathscr{S}^+(i)} & B_{k+1} \\ C_{\mathscr{K}-\mathscr{S}^+(i)} & 0 & 0 \end{bmatrix} \geqslant n+1 \quad \text{for all } \lambda \in C^+ - \Sigma_1 \quad (\Sigma_1 \text{ is finite}) \tag{23}$$

Combining (21) and (23), we have that both conditions are satisfied for $\lambda \in C^+ - (\Sigma \cup \Sigma_1)$.

Reasoning the same way as in the proof of Lemma 2, this implies that there exists a G such that

$$\rho\begin{bmatrix} \Delta(\lambda_0) - B_{k+1}GKC_{k+1} & B_{\mathscr{S}^+(i)} \\ C_{\mathscr{K}-\mathscr{S}^+(i)} & 0 \end{bmatrix} \geqslant n+1 \quad \text{for some } \lambda_0 \in C^+ - (\Sigma \cup \Sigma_1)$$

Moreover,

$$\hat{\mathscr{G}} = \left\{ G \in \mathbb{R}^{m_{k+1} \times h} \,\middle|\, \rho\begin{bmatrix} \Delta(\lambda_0) - B_{k+1}GKC_{k+1} & B_{\mathscr{S}^+(i)} \\ C_{\mathscr{K}-\mathscr{S}^+(i)} & 0 \end{bmatrix} \geqslant n+1 \right\}$$

is robust. Therefore, we can select a $G \in \tilde{\mathscr{G}} \cap \hat{\mathscr{G}}$ such that for all $\mathscr{S} \in \mathscr{P}(\mathscr{K})$

$$C_{\mathscr{K}-\mathscr{S}^+(i)}(\Delta(\lambda) - B_{k+1}GKC_{k+1})^{-1}B_{\mathscr{S}^+(i)} \neq 0$$

and

$$\rho\begin{bmatrix} \Delta(\lambda) - B_{k+1}GKC_{k+1} & B_{\mathscr{S}^+(i)} \\ C_{\mathscr{K}-\mathscr{S}^+(i)} & 0 \end{bmatrix} \geqslant n \quad \text{for all } \lambda \in C^+$$

Let $F_{k+1} = GK$. Since the theorem holds for $N = k$, we have that

$$\rho\left[\Delta(\lambda) - \sum_{\substack{j=1 \\ (j \neq i)}}^{k+1} B_j F_j C_j \quad B_i \right] = n \quad \text{for all } \lambda \in C^+$$

and hence the theorem holds for $N = k + 1$. □

Theorem 4

For system (1), there exist feedback matrices $F \in \mathbb{R}^{m_j \times p_j}$, $j = 1, 2, \ldots, N$ to make system (2) completely controllable and observable if and only if for all $\mathscr{S} \in \mathscr{P}(\mathscr{N})$ we have

$$C_{\mathcal{N}-\mathcal{S}}(\lambda I - A)^{-1}B_{\mathcal{S}} \neq 0 \tag{24}$$

and

$$\rho\begin{bmatrix} \lambda I - A & B_{\mathcal{S}} \\ C_{\mathcal{N}-\mathcal{S}} & 0 \end{bmatrix} \geqslant n \quad \text{for all } \lambda \in \sigma(A) \tag{25}$$

Proof

The proof is just a simple extension of the proof of Theorem 2 and the details are omitted.

Remark 2

Note that conditions (24) and (25) do not depend on the specific channel i. The important conclusion that can be drawn from this observation is that if system (1) can be made completely controllable and observable from channel i, then it can be made completely controllable and observable from any channel. This fact is also true for stabilizability and detectability.

Corollary 3

For system (1), assume that $C_{\mathcal{N}-\mathcal{S}}(\lambda I - A)^{-1}B_{\mathcal{S}} \neq 0$ for all $\mathcal{S} \in \mathcal{P}(\mathcal{N})$. Then there exist feedback matrices $F_j \in \mathbb{R}^{m_j \times p_j}, j = 1, 2, \ldots, N$, to make (2) stabilizable and detectable if and only if for all $\mathcal{S} \in \mathcal{P}(\mathcal{N})$,

$$\rho\begin{bmatrix} \lambda I - A & B_{\mathcal{S}} \\ C_{\mathcal{N}-\mathcal{S}} & 0 \end{bmatrix} \geqslant n \quad \text{for all } i \in C^{+} \cap \sigma(A)$$

Corollary 4

Assume that system (1) is strongly connected. Then system (1) is single-channel controllable (stabilizable) and observable (detectable) if and only if system (1) has no fixed modes (has fixed modes in C^{-}).

Proof

This result follows from Theorem 4 and Proposition 6. □

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